Basics of Special Relativity

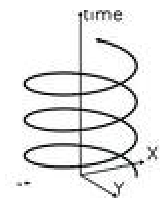
You must understand special relativity in order to really understand general relativity. Here's a brief summary of the basic ideas and terminology of special relativity (there is much more in GR2b) :

A **frame of reference** (or "reference frame") is a space (a laboratory, for example) which has a coordinate system that is *conceptually* built from measuring sticks and synchronized clocks all at rest relative to each other. Within this space, all measurements of physical processes will be self-consistent.

An **inertial frame** (or "inertial frame of reference", "inertial coordinate system", "inertial reference frame", "local Lorentz frame", etc.) is a frame of reference in which objects move at a constant velocity (including zero) unless acted on by an external force (external to the object, not the frame). An inertial frame can be moving at a constant velocity, but cannot be accelerating, changing directions, or rotating. No experiment can distinguish one inertial frame from another.

The three dimensions of space and one of time are combined into a single four-dimensional manifold called **spacetime**, which has a basis [ct, x, y, z]. While time (t) is the first element, it is converted into a distance by multiplying it by the speed of light (c). NOTE : some authors use [x, y, z, ct] as a basis. Special relativity assumes spacetime is flat (so all Euclidean relationships hold), which is called **Minkowski** spacetime.

The **worldline** of an object is the path it traces thru spacetime. The worldline of the earth (as seen from the sun), using *x* and *y* for space and *z* for time, looks like a corkscrew. → If you looked down on this figure from the *z* axis, you would see the earth moving around the sun.



A **four-vector** is a vector in spacetime. Four-vectors differ from "regular" vectors in that they can be altered by Lorentz transformations and not change length. Velocity, acceleration, momentum, force, and current are examples of four-vectors. Here, when using index notation with four-vectors in Cartesian coordinates, 0 = t, 1 = x, 2 = y, and 3 = z. In relativity, the Greek alphabet and the Roman alphabet are used to distinguish whether the index is summed over 1,2,3 or 0,1,2,3. Usually, Roman (i, j, k, ...) is used for 1,2,3 and Greek (α, β, μ, ν, ...) is used for 0,1,2,3. NOTE : some authors use 1,2,3,4 and/or reverse the roles of Roman and Greek letters, or ignore this convention.

Lorentz transformations include rotations in space and *boosts* (a change in velocity), and are described by 4×4 matrices/tensors. For example, the Lorentz transformation for a boost with a velocity *v* in the *x* direction is :

$$\begin{cases} t' &= \gamma\left(t - vx/c^2\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{cases} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

This can be written in vector/tensor form as :

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix} \qquad \text{where} \qquad \beta = v / c$$

Note that β is always ≤ 1 and γ is always ≥ 1.

An **event** is any point in spacetime at which something happens. The **displacement** four-vector is an "arrow" linking two events, and is described by the difference between them : $(c\Delta t, \Delta x, \Delta y, \Delta z)$. Special relativity always deals with displacement vectors (which are tensors of rank 1), as opposed to position vectors.

Einstein originally assumed two postulates (assumptions) as the starting point for special relativity :

The laws of physics are the same in any/every <u>inertial</u> frame
The speed of light is the same in any/every <u>inertial</u> frame

From the Lorentz transformations and Einstein's postulates come many "strange" results :

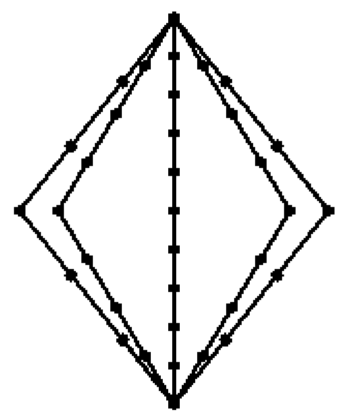
Relativistic mass – object A traveling relative to object B appears more massive to B
Time dilation – clock A traveling relative to clock B seems to run slower than clock B
Length contraction – object A traveling relative to object B seems shorter to B
Non-simultaneity – events that appear simultaneous to A may not appear simultaneous to B
(but if A says that event 1 *caused* event 2, B always will too)
Combining velocities – velocities near the speed of light do not just simply add
Mass/energy equivalence – the total energy of an object in motion is $E = (m^2c^4 + p^2c^2)^{1/2}$ where
m = rest mass of the object
$p = \gamma \cdot m \cdot v$ = relativistic momentum of the object
v = velocity of the object

The **spacetime interval** (or just "interval") Δs is the "distance" between two events, and is given by :

$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ NOTE : "Δs^2" (et al.) always means $(\Delta s)^2$ not $\Delta(s^2)$

While two observers in motion relative to each other may assign different locations and times to two events, they will both agree on the interval between the events. Thus the interval is *invariant* between frames.

Imagine a particle moving from A to B. There is exactly one <u>unaccelerated</u> worldline W connecting A and B (the vertical line in the figure to the right). Then in a reference frame traveling along W, the time between A and B is as <u>large</u> as it can be. This is because for an observer following W, his clock is always stationary, while clocks following any other worldline from A to B (the other lines) will be moving relative to him at least some of the time. Because moving clocks run slow, those observers will measure a smaller time between A and B than the observer following W will. The time interval measured in W (*which is the worldline of the particle*) is called the **proper time**. Proper time is *invariant* because it is a <u>property of the events</u>, not the reference frame ("proper" here means "belonging to", not "correct"). Proper time is represented by the symbol τ to distinguish it from **coordinate time** t which is the time between two events as measured by an observer using the clock in their inertial frame.



↑time A x →

<u>Free Fall</u>

Before we get to general relativity, the idea of "free fall" must be defined : an object is in **free fall** when its motion is affected by <u>no forces at all</u>, or <u>no forces except gravity</u> (no rockets on it, no electric or magnetic fields pulling it around, etc.).

Examples of objects in free fall :

- A space ship with its rockets off (in orbit or coasting thru space)
- A small, heavy object falling towards the Earth (with little wind resistance)
- A small, heavy object thrown upwards (with little wind resistance). Technically, an object is in free fall even when moving upwards or instantaneously at rest at the top of its motion, since the only force it is feeling is gravitational (downwards)

It is important to understand that free fall does not depend on whether there is a gravitational source nearby – a spaceship in orbit around a planet and one that is coasting thru interstellar space are both in free fall.

Examples of objects not in free fall :

- An accelerating space ship (firing its rockets)
- Flying in an aircraft – the force of gravity is balanced by a lift force
- Standing on the ground – the gravitational force downward is counteracted by the equal and opposite force up from the ground
- Falling towards the Earth using a parachute – there is a drag force which reduces the force of gravity
- A falling skydiver who has not yet opened their parachute is not considered in free fall by this definition, since they experience a drag force which equals their weight once they have achieved terminal velocity

In general relativity we can now define a **geodesic** *in spacetime* as the worldline of a free-falling object.

Basics of General Relativity

There are several essential ideas underlying the theory of general relativity. There seems to be a great deal of disagreement as to whether these ideas are "postulates", "equivalence principles", "theorems", or "results", and whether the equivalence principles are "strong" or "weak". So they are just presented here without titles.

All laws of physics must have the same form for observers using any coordinate system. For equations to be expressed in a form that is valid independent of any coordinate system, tensors (of all ranks) must be used. Such equations are called "covariant" (which has nothing to do with covariant vectors below).

Spacetime may be described as a curved (non-Euclidean), four-dimensional manifold. Differential geometry is the mathematics of curved manifolds.

All laws of physics must have the same form for observers in any frame of reference, whether inertial or accelerated. Any equation which uses tensors (of any rank) that is true in Minkowski spacetime must also be true in curved spacetime (as long as we replace "regular" derivatives with covariant derivatives).

It is impossible to distinguish between observations made while accelerating (in a spaceship with engines firing) **and those made at rest in a gravitational field** (standing on the surface of the Earth)**.** Some authors state this as : inertial (accelerated) mass is identical to gravitational mass. Others state it as : observations made in free fall in a gravitational field and those made far from any gravitational field are indistinguishable.

At every point in spacetime there exists a local inertial reference frame, corresponding to locally flat coordinates carried by a freely falling observer, in which the physics is indistinguishable from that of special relativity, at least for an instant. The local coordinates of any object in free fall are locally flat (because in the accelerated frame of reference the gravitational field appears to vanish*, which is why astronauts in orbit are "weightless"). "Locally flat" here means that a "small enough" region around the point looks like Minkowski spacetime. In locally flat coordinates, objects behave as they do in Newtonian mechanics : they move at a constant velocity (including zero) unless acted on by an external force.

*Technically, in a free falling local inertial frame all first order effects of gravity vanish, but second order effects (like tidal forces) remain. It is generally impossible to find an inertial frame in which all effects due to gravity vanish.

Mass and energy curve spacetime. This bending is usually very slight unless one is near a very large mass like the Sun or Earth. But this bending preserves the local smoothness of spacetime so that altho it is no longer flat, it is still a manifold and we can do all sorts of things with it (mathematically) as long as we restrict ourselves to a locally flat region.

In addition to the above ideas, some of which are already in direct contradiction to special relativity, there are some other significant differences between special and general relativity :

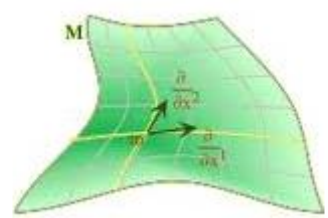
In special relativity, we cannot talk about absolute velocities, but only relative velocities. For example, we cannot sensibly ask if a particle is at rest, only whether it is at rest relative to another particle. **In general relativity, we cannot even talk about relative velocities**! This is because we can only unambiguously compare two velocities if they are at the same point in spacetime – at the same place at the same instant.

In general relativity, inertial frames are meaningless. In special relativity we can think of an inertial frame as being defined by a field of clocks, all at rest relative to each other. In general relativity this makes no sense, since we can only unambiguously define the relative velocities of two clocks if they are at the same location.

Gravity in general relativity is not a "force", just a manifestation of the curvature of spacetime. Suppose two people stand 100 feet apart on the equator of the Earth, and start walking north, both following geodesics. Although they start out walking parallel to each other, the distance between them gradually starts to shrink, until finally they bump into each other at the north pole. If they didn't understand the curved geometry of a sphere, they might think a "force" was pulling them together. Likewise, an object in free fall near a gravitational source (planet, sun) is falling because that is how an object moves when there is no force being exerted on it (so it is following the geodesic of the curved spacetime), instead of being due to the "force of gravity". In addition, all particles follow the same geodesic, regardless of the shape or composition of the particle.

Concepts and Notations

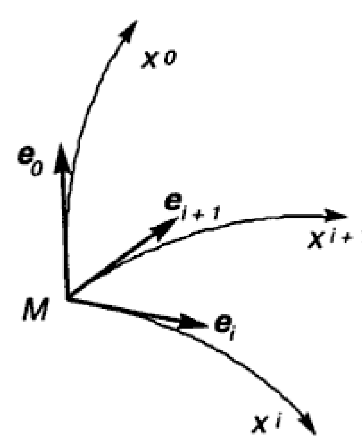
Before we go further, we need to know how the ideas from GR1a relate to general relativity and the formulas we're going to be using.



Since spacetime is curved, the coordinate system in the surface is *curvilinear*, even though it may look orthogonal to observers within the surface. →

Any x^i or x_i terms in formulas represent the curvilinear coordinates in the surface.

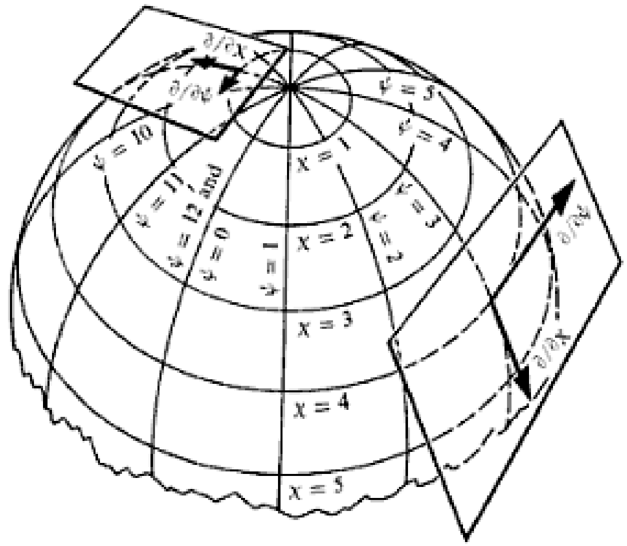
Tangent vectors are also called "basis vectors", "coordinate basis vectors", "local basis", or the "coordinate basis" by different authors. Here "**tangent vector**" will be used to refer to any vector that is tangent to a curved surface at a point P. But the term "**tangent basis vector**" will be used to refer specifically to vectors that are also tangent to (in-line with) the curvilinear coordinates at P. Tangent basis vectors are the *basis* for the *tangent plane*, which is a *locally flat spacetime* but with a (generally) *non-orthogonal* coordinate system. In the picture to the right, the x^i's are the curvilinear coordinates, and the $\mathbf{e}_i$'s are the tangent basis vectors. Tangent basis vectors contain the local information about the curvature of the space, because at P they are pointing in the same directions as the curvilinear coordinates. Keep in mind that tangent vectors are **not** "in" the surface!



If an equation has no x^i or x_i terms in it, the math is taking place in the tangent plane!

In any coordinate system, the tangent basis vectors $\mathbf{e}_a$ are tangent to the coordinate curves, so we can write the tangent basis vectors as derivatives of the coordinates : $\mathbf{e}_a = \partial_a = \partial/\partial x^a = \partial P/\partial x^u$, where the point P is defined in the curvilinear coordinates.

In the picture to the right, two tangent planes are shown on the surface of a sphere, with their tangent basis vectors…well, tangent…to the lines of latitude and longitude (the curvilinear coordinates of the surface). The orientations of the tangent planes are defined by the tangent basis vectors.



For a sphere with a <u>fixed radius</u> *r* (so it is not a coordinate), we can parameterize the surface by using $x^1=\theta$ and $x^2=\varphi$, so that the Cartesian vector for P (in terms of the curvilinear coordinates) is

$$P = (r \sin\theta \cos\varphi,\ r \sin\theta \sin\varphi,\ r \cos\theta)$$

Then the tangent basis vectors are given by $\partial P/\partial x^u$ or

$$\partial P/\partial x^1 = \partial P/\partial\theta = (r \cos\theta \cos\varphi,\ r \cos\theta \sin\varphi,\ -r \sin\theta)$$
$$\partial P/\partial x^2 = \partial P/\partial\varphi = (-r \sin\theta \sin\varphi,\ r \sin\theta \cos\varphi,\ 0)$$

The point P for the tangent plane on the right side of the figure is at θ=60° and φ=90° (which is in the *yz* plane), and assuming r = 1 then :

$\partial P/\partial\theta = (0, 0.5, -0.866)$ is the downward-pointing arrow

$\partial P/\partial\varphi = (-0.866, 0, 0)$ is the backward-pointing arrow

<u>The Metric Tensor and the Minkowski Metric</u>

The “metric tensor” (or just “metric”) is the centerpiece of general relativity. It describes everything about the geometry of spacetime, since it lets us measure angles and distances. Einstein's equation describes how the flow of matter and energy through spacetime affects the **curvature** of the metric. Everything in general relativity is related to the curvature!

The value of the metric tensor g_{ab} at point P is defined as the dot product of the *tangent basis vectors* at P. Thus the value of the metric depends on the shape of the surface, and so contains information about its curvature. The metric is *defined by* the tangent space, but <u>locally</u> *describes* the curved space.

The metric can also be thought of as a function that takes a pair of any tangent vectors **v** and **w**, and produces a scalar in a way that generalizes many properties of the vector dot product in Euclidean space. In the same way as the dot product, the metric tensor is used to calculate the length of, and angle between, tangent vectors. It is called the “metric” tensor because it defines the way length is measured.

In general, the inner product or “dot product” of two tangent vectors **v** and **w** is :

$$g(\mathbf{v},\mathbf{w}) = g_{ab}\, v^a\, w^b = \Sigma_a \Sigma_b\, g_{ab}\, v^a\, w^b = g_{00}v^0w^0 + g_{01}v^0w^1 + g_{02}v^0w^2 + g_{03}v^0w^3 + g_{10}v^1w^0 + \ldots$$

In the odd geometry of curved spacetime it is not obvious what *perpendicular* means. We therefore *define* two four-vectors **v** and **w** to be perpendicular if their dot product is zero, the same as three-vectors : $\mathbf{v}\cdot\mathbf{w}=g(\mathbf{v},\mathbf{w})=0$. Since the inner product is coordinate invariant (it is a tensor of rank 0), 4-vectors that are perpendicular in one frame are perpendicular in all frames.

The metric tensor can be used to compute the distance between any two points in a curved space, based on the *difference* dx^i between them :

$$ds^2 = g_{ab}\, dx^a\, dx^b$$

which can then be integrated to get the distance.

This last formula is called the **line element**, and plays a very important role in general relativity. Once you know the metric you have the line element (and vice-versa), and quite often descriptions of a specific spacetime are given by the line element but are *called* "the metric".

In a 2-D Euclidean space with Cartesian coordinates, $g_{ab} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

and so the line element is :

$$ds^2 = 1{*}dx_1{*}dx_1 + 0{*}dx_1dx_2 + 0{*}dx_2dx_1 + 1{*}dx_2{*}dx_2 = dx_1^2 + dx_2^2 = dx^2 + dy^2$$

which is the Pythagorean theorem.

The line element in 2-D Euclidean space with polar coordinates is :

$$ds^2 = dr^2 + r^2d\theta^2 = dx_1^2 + x_1^2dx_2^2$$

so the metric must be : $g_{ab} = \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix}$

Note that the presence of the r^2 does not mean the space is curved, it is just "part of" the coordinate system.

Likewise, the line element in 3-D Euclidean space with spherical coordinates is :

$$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

The metric can also calculate the length *s* of a tangent vector **v** :

$$s^2 = g_{ab}\, v^a\, v^b$$

As mentioned before, the metric tensor can be calculated from the basis vectors $\mathbf{e}_i$ by

$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ for all (i,j) – note there is no summation!

where $\cdot$ is the standard vector "dot" product, and $\mathbf{e}_i$ is the i^{th} *basis vector*.

The metric tensor is always symmetrical, meaning that $g_{ab} = g_{ba}$.

The inverse of the metric tensor is written g^{ab} and is the matrix inverse of g_{ab}.

The metric of any Euclidean space with orthogonal coordinates is diagonal and so

$$g^{ab} = 1 / g_{ab}$$

which means to take the reciprocal of each non-zero element of g_{ab} individually (no summations).

In flat 4-D spacetime (special relativity), with Cartesian coordinates, the metric tensor is often called the Minkowski metric η (assuming units where c = 1) :

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

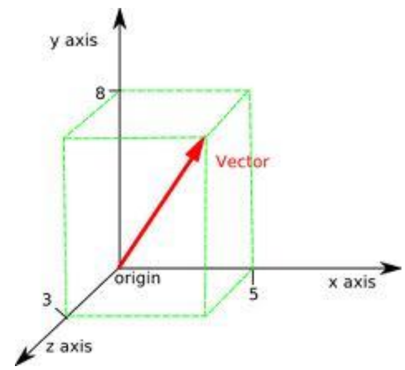
NOTE : some authors define η with the opposite sign :

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If the value of the metric tensor $g^{ab} = \eta^{ab}$ at a particular point, the spacetime is locally flat there.

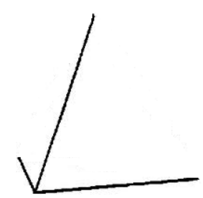
Covariant and Contravariant Vectors and Tensors

The terms "covariant" and "contravariant" merely refer to two different ways of defining **the values of the coordinates** of a tensor of any rank. And in Euclidean space with Cartesian coordinates, the covariant and contravariant descriptions are identical. Covariant coordinates go by many other names, including "one-form", "reciprocal basis", "covector", and "dual". Contravariant and covariant vectors are discussed in much more detail in GR2a and especially in GR2d.



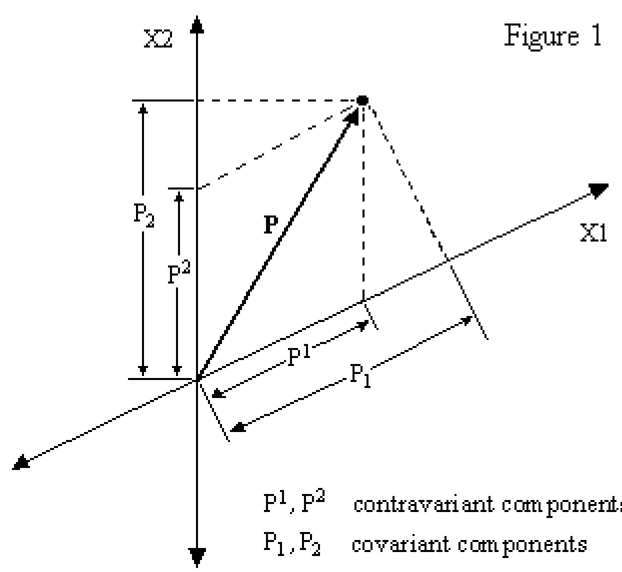
At any point P in 3-D Cartesian coordinates, we can specify three local axes which determine three local planes perpendicular to each axis. Since they are Cartesian, the axes must be mutually perpendicular and so must the planes. Now, choose three unit vectors at P such that each vector is *tangent to* (in-line with) an axis. → Such a basis can be designated [**i**, **j**, **k**]. Any vector **V** at P can then be written **V** = a**i** + b**j** + c**k** or **V** = (a, b, c) where a, b, and c are the usual scalar components of the vector.

Now suppose that we had chosen unit vectors *perpendicular to each of the planes* rather than *tangent to each of the coordinate axes*. Let's call the resulting basis [**i***, **j***, **k***]. Again, any vector **V** at P can be written **V** = a***i*** + b***j*** + c***k*** where a*, b*, and c* are the scalar components of the vector using the basis [**i***, **j***, **k***]. There is nothing wrong with this provided we ensure that a**i** + b**j** + c**k** = a***i*** + b***j*** + c***k***. This may seem trivial since it is apparent from the geometry that the two bases are the same because **i** = **i*, j** = **j*, k** = **k*.**



Still, we used two *distinct* approaches to defining a basis at P, which in general will not give the same results. Let's modify our axes so they are *no longer mutually orthogonal* – for example, so that they meet at 60°. In this case, the origin lies at a vertex of a tetrahedron, and the axes lie along three of the edges. → It should be obvious that (**i**, **j**, **k**) and (**i***, **j***, **k***) are now two *different* sets of unit vectors. Specifically, **i** and **i*** now meet at an angle of 60°, as do **j** and **j***, and **k** and **k***. Thus, while they are all unit vectors, they specify *different sets of directions*.

The same logic can be applied to curvilinear coordinate systems. At any point P in curvilinear coordinates (with its set of local axes and coordinate surfaces), we can specify two related but distinct sets of unit vectors : (1) a basis *tangent to the local axes*, called **contravariant**, and (2) another basis *perpendicular to the local coordinate surfaces*, called **covariant**. The vector **V** can be referred to in terms of either basis, and the vector itself is obviously not affected by the choice.



To visualize this better, let's look at some 2-D examples. The figure to the right shows a non-orthogonal coordinate system with axes X1 and X2, and the contravariant and covariant components of the position vector **P** with respect to these coordinates. *The covariant/reciprocal axes (not shown) are perpendicular to X1 and X2.*

As can be seen, the i^{th} **contravariant** component consists of the projection of **P** onto the i^{th} axis in a direction parallel to the other axis ("tangent to the axes"), whereas the i^{th} **covariant** component consists of the projection of **P** in a direction perpendicular to the i^{th} axis ("perpendicular to the surfaces").

Contravariant <u>components</u> always have a superscript (P^1,P^2), while covariant components always have a subscript (P_1,P_2). But just to make things confusing, contravariant <u>bases</u> have subscripts [$\mathbf{e}_1$,$\mathbf{e}_2$], while covariant bases have superscripts [$\mathbf{e}^1$,$\mathbf{e}^2$]. Actually, this is done so that the Einstein summation notation works : a contravariant vector is described by $P^i\mathbf{e}_i = P^1\mathbf{e}_1+P^2\mathbf{e}_2$, and a covariant vector is described by $P_i\mathbf{e}^i = P_1\mathbf{e}^1+P_2\mathbf{e}^2$. Also, contravariant vectors are represented as column vectors while covariant vectors are row vectors.

In Euclidean space with Cartesian coordinates, the contravariant and covariant interpretations are identical. This can be seen by imagining that the coordinate axes in the figure above are perpendicular to each other. But even when the space is flat, if the coordinates are *curvilinear* (polar, spherical) the contravariant and covariant descriptions are <u>different</u>.

In a space with a metric, the line interval in terms of the contravariant components is :

$$ds^2 = g_{uv}\, dx^u\, dx^v$$

And in terms of the covariant components as :

$$ds^2 = g^{uv}\, dx_u\, dx_v$$

Consider a vector d**x** whose contravariant components (dx^1, dx^2) are multiplied by the covariant metric tensor :

$$g_{uv}dx^u = dx_v$$

Treat the upper and lower *u*'s as "cancelling" each other, leaving the *v* to be attached to the dx term.

Also note that order does not matter : $x^u g_{uv} = g_{uv}x^u$ because of the Einstein notation – for example,

$$x_2 = x^u g_{u2} = \Sigma_i\, (x^i \cdot g_{i2}) = x^1{\cdot}g_{12} + x^2{\cdot}g_{22} = g_{12}{\cdot}x^1 + g_{22}{\cdot}x^2 = \Sigma_i\, (g_{i2} \cdot x^i) = g_{u2}x^u$$

Likewise :

$$g^{uv}dx_u = dx^v$$

Hence we can convert from the contravariant to the covariant versions of a vector by multiplying by the metric tensor, and we can convert back by multiplying by the inverse of the metric tensor. These operations are called "raising and lowering the indices", because they convert **x** from a superscripted to a subscripted variable, or vice versa. In this way we can also create mixed tensors (of rank 2 or more), that are contravariant in some of their indices and covariant in others.

It's worth noting that since $dx_u = g_{uv}\, dx^u$ we have

$$ds^2 = (g_{uv}\, dx^u)\, dx^v = dx_u dx^u$$

<u>An example with numbers</u> :
Let's assume we have a vector **P** = (5,12) in a standard Cartesian system using unit vectors <1,0> and <0,1>.

Now let's define a non-orthonormal basis :

$\mathbf{A}_1$ = <0.50, 0.0>
$\mathbf{A}_2$ = <0.75, 0.5>

We will call this our <u>contravariant</u> basis. Then we can show that

$\mathbf{P} = -26{*}\mathbf{A}_1 + 24{*}\mathbf{A}_2$ because -26*(0.50, 0.0) + 24*(0.75, 0.5) = (-13,0)+(18,12) = (5,12)

So our contravariant components of P are :

$P^1 = -26$ and $P^2 = 24$

Now let's define another basis that is reciprocal to $[A_1, A_2]$ and call it $[A^1, A^2]$:

$\mathbf{A}^1$ = <2.0, -3.0>

$\mathbf{A}^2$ = <0.0, 2.0>

Notice that the Euclidean inner product between these two bases is the identity matrix, so they are reciprocals :

$\mathbf{A}^1 \cdot \mathbf{A}_1 = 1 \qquad \mathbf{A}^1 \cdot \mathbf{A}_2 = 0$

$\mathbf{A}^2 \cdot \mathbf{A}_1 = 0 \qquad \mathbf{A}^2 \cdot \mathbf{A}_2 = 1$

In this covariant basis we can show that

$\mathbf{P} = 2.5*\mathbf{A}^1 + 9.75*\mathbf{A}^2$ because (5,-7.5) + (0,19.5) = (5,12)

Our covariant components of P are now :

$P_1 = 2.50$ and $P_2 = 9.75$

In all cases, **P** is still the same vector **P**, we have just described it in two different bases (really three if you include the Cartesian description) :

	Cartesian		Contravariant		Covariant
P =	(5,12)	=	(-26, 24)	=	(2.5, 9.75)

In the Cartesian case the metric components are:

g_{11} = <1,0> · <1,0> = 1

g_{12} = <1,0> · <0,1> = 0

g_{21} = <0,1> · <1,0> = 0

g_{22} = <0,1> · <0,1> = 1

$$\mathbf{g} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

In the contravariant case the metric components are :

$g_{11} = \mathbf{A}_1 \cdot \mathbf{A}_1$ = <0.50, 0.0> · <0.50, 0.0> = 0.2500

$g_{12} = \mathbf{A}_1 \cdot \mathbf{A}_2$ = <0.50, 0.0> · <0.75, 0.5> = 0.3750

$g_{21} = \mathbf{A}_2 \cdot \mathbf{A}_1$ = <0.75, 0.5> · <0.50, 0.0> = 0.3750

$g_{22} = \mathbf{A}_2 \cdot \mathbf{A}_2$ = <0.75, 0.5> · <0.75, 0.5> = 0.8125

$$\mathbf{g} = \begin{vmatrix} .25 & .375 \\ .375 & .8125 \end{vmatrix}$$

In the covariant case the metric components are :

$g^{11} = \mathbf{A}^1 \cdot \mathbf{A}^1$ = <2, -3> · <2, -3> = 13

$g^{12} = \mathbf{A}^1 \cdot \mathbf{A}^2$ = <2, -3> · <0, 2> = -6

$g^{21} = \mathbf{A}^2 \cdot \mathbf{A}^1$ = <0, 2> · <2, -3> = -6

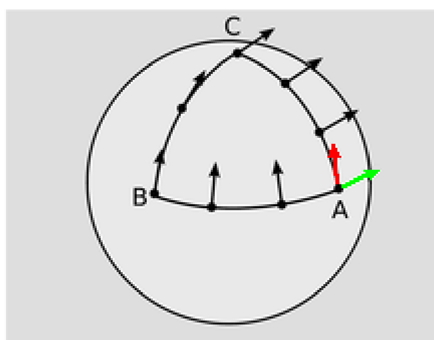
$g^{22} = \mathbf{A}^2 \cdot \mathbf{A}^2$ = <0, 2> · <0, 2> = 4

$$\mathbf{g} = \begin{vmatrix} 13 & -6 \\ -6 & 4 \end{vmatrix}$$

Notice that $g_{ij} = g_{ji}$ and $g_{ij} = (g^{ij})^{-1}$ (trust me, it is) as they should be.

Parallel Transport

The idea here is that our "space" is the 2-dimensional *surface* of the earth, and a javelin is the "tangent vector", which must remain tangent to the space. This means it must always be held horizontally during all that follows. You are also instructed to always keep the javelin pointing "in the same direction that it has been pointing".



At A, you start out with the javelin pointing north (the red vector). Pointing the javelin to your right, you march 90 degrees west around the equator, where at B you turn right, but keep the javelin pointing in the same direction it has been pointing. Now holding the javelin straight ahead, you march north till you reach the north pole, where at C you turn right, and keep the javelin pointing in the same direction it has been pointing. So holding the javelin to your left, now you march south until you reach your starting point. That's "parallel transport".

But wait! When you arrive back at A, the javelin is now the green vector, and isn't pointing in the same direction as when you started! That's because the surface is curved, which is explained more in GR1c. Your job here was to do the parallel transport, which you did.

By the way, we now have yet *another* definition for a geodesic : a geodesic curve is one that parallel transports any tangent vector without changing its direction or length. Each segment AB, BC, CA in the example is a geodesic (part of a great circle), and notice that the javelin never changed its direction along the segment.

Parallel transport is path-dependent, even if you follow geodesics : if the vector v_0 at A in the figure below is transported thru B to D, the result is v_1. But if v_0 is transported thru C to D, the result is v_2.