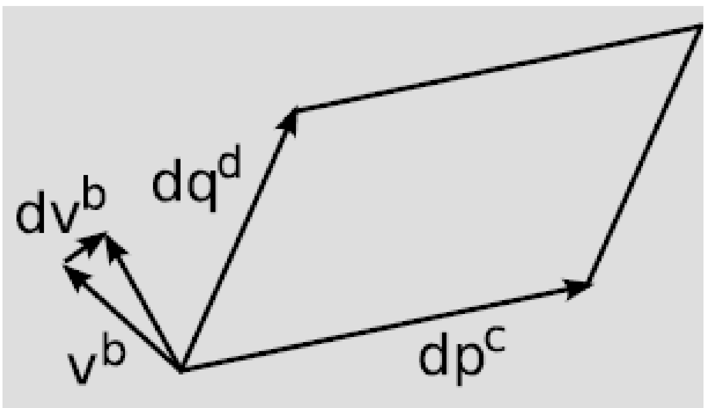
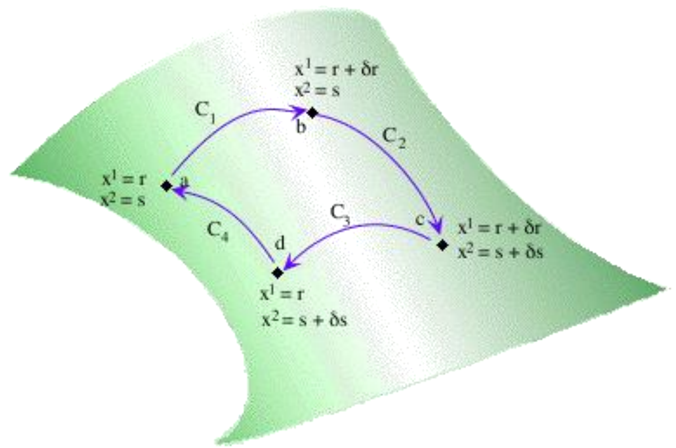
Riemann Curvature Tensor

Almost everything in Einstein's equation is derived from the Riemann tensor ("Riemann curvature", "curvature tensor", or sometimes just "the curvature").

The Riemann tensor $R^a{}_{bcd}$ is a tensor that takes three tangent vectors (say **u**, **v**, and **w**) as inputs, and outputs one tangent vector, R(**u**,**v**,**w**). What the Riemann tensor does is this :

Take the vector **w**, and parallel transport it around a small parallelogram whose two edges point in the directions ε**u** and ε**v** , where ε is a small number.





The vector **w** comes back a bit changed by its journey (as you saw in "Parallel Transport" in GR1b) – it is now a new vector **w**'. The amount of change is proportional to the <u>area</u> traveled (ε^2). We then have

$$R(\mathbf{u},\mathbf{v},\mathbf{w}) \approx (\mathbf{w} - \mathbf{w}') / \varepsilon^2$$

Thus the Riemann tensor keeps track of how much a parallel transport around a small parallelogram changes the vector **w**. When we say "spacetime is curved", this is *identical* to saying that "parallel transport around a loop changes a vector". <u>All</u> the information about the curvature of spacetime is contained in the Riemann tensor!

We can also describe the function of the Riemann tensor using index notation :

$$R^a(\mathbf{u},\mathbf{v},\mathbf{w}) = R^a{}_{bcd}\, u^b\, v^c\, w^d$$

So a <u>specific</u> element $R^a{}_{bcd}$ of the Riemann tensor tells us how much a vector pointing in the *d* direction swings over towards the *a* direction when we parallel transport it around a little parallelogram in the *b*-*c* plane.

NOTE : some authors use a different version of the Riemann tensor R_{abcd}, but $R^a{}_{bcd} = g^{aa} R_{abcd}$

Ricci Tensor

The Ricci tensor is defined as

$$R_{bd} = R^i{}_{bid} = R^0{}_{b0d} + R^1{}_{b1d} + R^2{}_{b2d} + R^3{}_{b3d}$$

The physical significance of the Ricci tensor is best explained by an example. Suppose an astronaut taking a spacewalk accidentally spills a can of ground coffee. Say that a given moment one coffee ground is at point P, and its velocity vector is the tangent four-vector **v**. The path the coffee ground traces out in spacetime is its worldline. Let's draw a little bit of its worldline near P :

```
v ^
  |
  |
  P
```

The vector **v** is an arrow with its tail at P, pointing straight up.

Now imagine a bunch of coffee grounds right near our original one, that are all initially at rest relative to it - or **comoving**. This means that for any tangent vector **w** at P which is orthogonal to **v**, if we follow a geodesic along **w** for a while, we find ourselves at a point Q where there's another coffee ground. Let's add the worldline of this other coffee ground :

```
v ^         ^ v'
  |         |
  | w       |
  P->-----Q
```

This shows that **w** is orthogonal to the worldline of our first coffee ground. The horizontal path is a geodesic from P to Q. The right-hand vertical line is the worldline of the coffee ground which goes through the point Q of spacetime, with velocity vector **v**'.

What does it mean to say the two coffee grounds are initially comoving? If we take **v** and parallel transport it over to Q along the horizontal path, we get **v**' (no matter how curved spacetime is in between them). This may seem like a lot of work to say that two coffee grounds are moving in the same direction at the same speed, but when spacetime is curved we must be careful how we describe things. Note that everything is based on parallel transport!

Now consider a little round ball of many comoving coffee grounds in free fall in outer space. If spacetime is flat, these coffee grounds would *stay* comoving as time passed. But if there is matter or energy nearby, spacetime will *not* be flat, and the coffee grounds will start moving relative to each other. As time passes, this ball will change shape and size depending on how the paths of the coffee grounds are deflected by the spacetime curvature. The ball may shrink or expand, rotate, twist, and/or get deformed into an ellipsoid. All the information about the curvature of spacetime is encoded in the rate at which this ball changes shape and size. But the Ricci tensor only keeps track of the rate of change of its volume, because it captures only some of the information in the Riemann curvature tensor – the rest is captured by something called the “Weyl tensor” (see the next section), which describes how the ball changes shape.

More precisely : the “speed” (first time derivative) at which the volume V of this little ball is changing is zero at the beginning, since the coffee grounds started out comoving. But the “acceleration” (second time derivative) at which the volume is changing is

$$d^2V/dt^2 \approx - V * R_{ab}v^a v^b$$

Weyl tensor

In 4-D spacetime, the Riemann tensor has 20 independent components. 10 of these are captured by the Ricci tensor, while the remaining 10 are captured by the Weyl tensor. The Weyl tensor is discussed in more detail in GR2f.

Recall the definition of the Ricci tensor in terms of a small round ball of coffee grounds floating through outer space, and what happens as time passes. Each coffee ground moves along a geodesic, but if spacetime is curved, the ball may rotate, twist, and/or be deformed into an ellipsoid. The Weyl tensor describes this part.

In general relativity, the Weyl curvature is the part of the curvature that exists in free space (which means it is a solution of the Einstein equation in a vacuum), and it governs the propagation of gravitational radiation, tidal forces, etc. through regions of spacetime devoid of matter and energy.

In 2-D and 3-D spaces the Weyl curvature tensor is always zero. In spaces with >3 dimensions, the Weyl curvature is generally nonzero. If both the Ricci tensor and the Weyl tensor are zero (in spaces with >3 dimensions), then spacetime is locally flat, which means there exists a local coordinate system in which the metric tensor is equal to the Minkowski metric.

The Einstein equation (which uses the Ricci tensor) only contains 10 pieces of information, although you need 20 to specify the Riemann curvature tensor. So the Einstein equation by itself doesn't let you reconstruct the complete curvature tensor. Why is this?

In electromagnetism, if you know the charge and current distributions everywhere in a volume of space, you might think that you can figure out the electric and magnetic fields everywhere in the volume. But you can't always – there's more information needed beyond just what the sources of the fields tell you. For example, there could be an electromagnetic wave passing thru. It doesn't have any source within the region you're considering, but it still affects the results in that region.

Likewise, in general relativity, knowing all about the sources (the stress-energy-momentum tensor **T**, discussed below) isn't enough to tell you all about the curvature. In both cases, you supplement the information from the sources with some extra initial conditions (like the presence of an independent electromagnetic field) to get a final solution. So even though it isn't used in Einstein's equation, the Weyl tensor carries information about the kind of curvature that is independent of the sources (for example, gravitational radiation passing by).

When we are in truly empty space, there's no Ricci curvature, so our ball of coffee grounds doesn't change volume. But there can be Weyl curvature due to gravitational waves, tidal forces, etc. from nearby masses which tend to stretch things out in one direction while squashing them in the other, without changing the total volume. When a ball of coffee grounds falls freely through outer space in the earth's gravitational field (in other words, is in orbit or is falling towards the Earth), it feels no Ricci curvature (because there is no large mass within the ball), only Weyl curvature from the nearby Earth. So the tidal forces due to some coffee grounds being nearer to the earth than others will stretch the ball into an ellipsoid.

Ricci Scalar

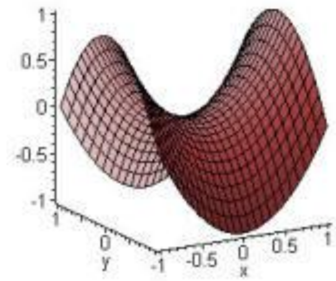
Given the Ricci tensor, then we define the Ricci scalar (or "scalar curvature") by

$$R = g^{ab} R_{ab} = R^a{}_a$$

$$\begin{aligned} &= g^{00}R_{00} + g^{01}R_{01} + g^{02}R_{02} + g^{03}R_{03} \\ &+ g^{10}R_{10} + g^{11}R_{11} + g^{12}R_{12} + g^{13}R_{13} \\ &+ g^{20}R_{20} + g^{21}R_{21} + g^{22}R_{22} + g^{23}R_{23} \\ &+ g^{30}R_{30} + g^{31}R_{31} + g^{32}R_{32} + g^{33}R_{33} \end{aligned}$$

This is a single number which basically represents the average overall curvature at any point in spacetime.

When the Ricci scalar curvature is positive at a point, the surface looks locally like the surface of a sphere or ellipsoid there. If it is negative, it looks like a hyperboloid or "saddle". →

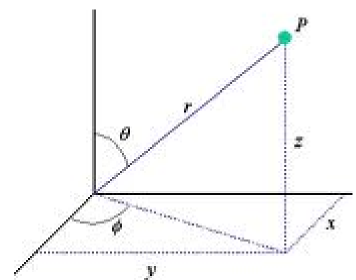


Also, when the Ricci scalar curvature is positive at a point, the volume of a small ball about the point has smaller volume than a ball of the same radius in Euclidean space. On the other hand, when the Ricci scalar curvature is negative at a point, the volume of a small ball is larger than it would be in Euclidean space.

Example : Curvature of a Sphere

In a 2-D circle it only takes *one number* to describe the Riemann curvature at every point, so there is the same amount of information in the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar. So we can't understand the differences between them very well with a circle.

For a 3-D sphere, which has a 2-dimensional surface, first we need some coordinates (see the figure to the right). Let's use the usual spherical coordinates θ and φ, where φ is the east-west angle from 0 to 2π, and θ is the angle from the north pole, going from 0 to π. The radius *r* is fixed, so it is not considered a coordinate.



So when we write something like g_{ab} or $R^a{}_{bcd}$, the index letters will go from 1 to 2, with "1" corresponding to θ and "2" corresponding to φ (and no summation!).

Then the metric tensor and its inverse are :

$$\begin{matrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{matrix} = \begin{vmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{vmatrix} \qquad \begin{matrix} g^{\theta\theta} & g^{\theta\varphi} \\ g^{\varphi\theta} & g^{\varphi\varphi} \end{matrix} = \begin{vmatrix} 1/r^2 & 0 \\ 0 & 1/(r^2 \sin^2\theta) \end{vmatrix}$$

The only non-zero elements of the Riemann curvature [a 2x2x2x2 tensor = 16 elements!] are :

$$R_{\theta\varphi\theta\varphi} = R_{\varphi\theta\varphi\theta} = r^2 \sin^2\theta$$

and

$$R_{\theta\varphi\varphi\theta} = R_{\varphi\theta\theta\varphi} = -\, r^2 \sin^2\theta$$

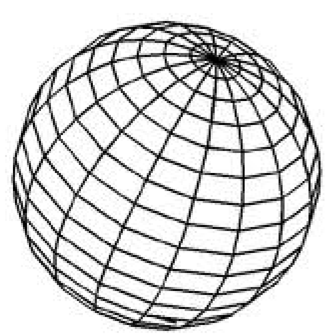
Then

$$R^\theta{}_{\varphi\theta\varphi} = g^{\theta\theta} R_{\theta\varphi\theta\varphi} = (1/r^2)(r^2 \sin^2\theta) = \sin^2\theta$$

and

$$R^\varphi{}_{\theta\varphi\theta} = g^{\varphi\varphi} R_{\varphi\theta\varphi\theta} = (1/r^2 \sin^2\theta)(r^2 \sin^2\theta) = 1$$

For example, the $R^\varphi{}_{\theta\varphi\theta}$ element is saying : take a tangent vector pointing in the θ direction, and carry it around a little square in the θ-φ plane. This is just parallel transport : we go in the θ direction a small amount ε, then we go in the φ direction until it has changed by ε, then we go back in the θ direction until the θ coordinate is what it started out as, and then we go back in the φ direction until the φ coordinate is what it started out as. Our original vector may have rotated a little bit since space is curved, which means that its component in the φ direction may have changed a bit. So $R^\varphi{}_{\theta\varphi\theta}$ tells us how much a vector in the θ direction swings over in the φ direction when we parallel transport it around a little square in the θ-φ plane (which is any box in the picture to the right). Note that the amount of change for $R^\varphi{}_{\theta\varphi\theta}$ is the same everywhere, but $R^\theta{}_{\varphi\theta\varphi}$ depends on your north-south position.



And since $R_{ij} = R^k{}_{ikj}$:

$$R_{\theta\theta} = R^\theta{}_{\theta\theta\theta} + R^\varphi{}_{\theta\varphi\theta} = 0 + 1 = 1$$
$$R_{\varphi\varphi} = R^\theta{}_{\varphi\theta\varphi} + R^\varphi{}_{\varphi\varphi\varphi} = \sin^2\theta + 0 = \sin^2\theta$$

So the Ricci tensor is :

$$\begin{matrix} R_{\theta\theta} & R_{\theta\varphi} \\ R_{\varphi\theta} & R_{\varphi\varphi} \end{matrix} = \begin{vmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{vmatrix}$$

When $\theta = 90^o$, the Ricci tensor is the identity matrix (which means a square at the equator is not stretched in any direction), and as θ approaches 0^o, the $\varphi\varphi$ (east-west) component gets very small (so the square gets distorted into a skinny wedge), and the "volume" (area) decreases. This agrees with the picture above. Also note that the values of the Ricci tensor do not depend on the radius of the sphere!

And finally, the Ricci scalar is

$$R = g^{ab} R_{ab} = g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi} = (1/r^2)(1) + (1/r^2\sin^2\theta)(\sin^2\theta) = 1/r^2 + 1/r^2 = 2 / r^2$$

Which says that the "overall" amount of curvature of a sphere does not change as you move anywhere around the sphere (which makes sense, otherwise it would be an ellipsoid), but it does change as the size of the sphere changes.

While we're here, what else can we do? Remember that g_{ab} lets us calculate the "dot product" of vectors by

$$g(\mathbf{v},\mathbf{w}) = g_{ab}\, v^a\, w^b = g_{11}v^1w^1 + \underbrace{g_{12}v^2w^2}_{=0} + \underbrace{g_{21}v^1w^1}_{=0} + g_{22}v^2w^2$$

So if we pick the vector **v** to be (0,1) where

$0 = v^1$ = length in the θ <u>direction</u> (north and south)

$1 = v^2$ = length in the φ <u>direction</u> (east and west)

Then the length of **v** is :

$$s = \sqrt{g(\mathbf{v},\mathbf{v})} = (g_{11}v^1v^1 + g_{22}v^2v^2)^{1/2} = (r^2{*}0{*}0 + r^2 \sin^2\theta{*}1{*}1)^{1/2} = r\cdot\sin\theta$$

Now **v** = (0, 1) is just a line segment parallel to the equator. As you can see from the picture above, its length should get shorter as it approaches the north pole, where $\varphi = 0$ and $\sin\varphi = 0$, which it does.

So, what does this entire example have to do with anything "real"? Nothing! It's just that we can visualize this space easily, and the math is simple.

<u>The Stress-Energy-Momentum Tensor</u>

We have mentioned several times that spacetime is curved by the matter and energy in it. How do we describe the distribution of matter and energy within a region of spacetime? With a tensor! The tensor $T_{\mu\upsilon}$ is called (among other things) the "stress-energy" or "energy-momentum" tensor, and represents various aspects of the matter and energy that are in the volume of space we're looking at :

$T_{\mu\upsilon}$ =

T_{00}	T_{01}	T_{02}	T_{03}
T_{10}	T_{11}	T_{12}	T_{13}
T_{20}	T_{21}	T_{22}	T_{23}
T_{30}	T_{31}	T_{32}	T_{33}

The T_{00} term is the *energy density* (how much total energy there is in a given volume of space at an instant in time). But keep in mind that matter and energy are equivalent in relativity ($E=mc^2$), so this term represents not only the electromagnetic energy (light, heat, radio, x-rays, etc.) in the volume, but how much matter there is (energy of matter = mc^2), and how fast it is moving (its kinetic energy).

The T_{i0} left-hand column (excluding T_{00}) describes how much mass-energy is *moving thru* the volume at a given instant.

The T_{0j} top row (excluding T_{00}) describes how much mass-energy is *moving thru* the volume *over* time.

The [T_{11} T_{22} T_{33}] terms represent the *pressure* (forces applied perpendicular to the surfaces of the volume, like the air that keeps a balloon blown up) being created *by* the matter or energy in the volume.

The remaining off-diagonal terms (in black) represent the *shear stress* (forces applied parallel to the surfaces of the volume) being created *by* the matter or energy in the volume.

In empty space, every component of $T_{\mu\upsilon}$ is zero.

For non-moving "dust" (small particles of matter that only interact gravitationally) :

$T_{\mu\upsilon}$ =

ρc^2	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

where ρ = mass density (and so the energy density comes from $E=mc^2$)

For a "perfect fluid" (no viscosity, no shear stresses, and no heat conduction) :

$T_{\mu\upsilon}$ =

ρc^2	0	0	0
0	P	0	0
0	0	P	0
0	0	0	P

ρ = mass density (as above)
P = pressure exerted by the fluid

The stress-energy-momentum tensor is discussed in much more detail in GR2f.

Einstein's Equation

Finally (!) we have the background needed to recognize all the terms in Einstein's equation for general relativity, which describes how matter and energy curve the manifold of spacetime :

$$R_{\mu\upsilon} - \tfrac{1}{2} g_{\mu\upsilon}\cdot R + \Lambda\cdot g_{\mu\upsilon} = k\cdot T_{\mu\upsilon}$$

Where

$R_{\mu\upsilon}$ = the Ricci tensor
$g_{\mu\upsilon}$ = the metric tensor
R = the Ricci scalar
Λ = a scalar representing a general expansion or contraction of spacetime (could be +, –, or 0)
k = a constant, the value of which depends on the system of units used : ±1, 8, 8π, $8\pi G$, $8\pi G/c^4$, etc.
$T_{\mu\upsilon}$ = the Stress-Energy-Momentum tensor

Keep in mind this formula actually represents 16 different equations for $\mu=(t,x,y,z)$ and $\upsilon=(t,x,y,z)$:

$R_{tt}-½g_{tt}R+g_{tt}\Lambda=kT_{tt}$	$R_{tx}-½g_{tx}R+g_{tx}\Lambda=kT_{tx}$	$R_{ty}-½g_{ty}R+g_{ty}\Lambda=kT_{ty}$	$R_{tz}-½g_{tz}R+g_{tz}\Lambda=kT_{tz}$
$R_{xt}-½g_{xt}R+g_{xt}\Lambda=kT_{xt}$	$R_{xx}-½g_{xx}R+g_{xx}\Lambda=kT_{xx}$	$R_{xy}-½g_{xy}R+g_{xy}\Lambda=kT_{xy}$	$R_{xz}-½g_{xz}R+g_{xz}\Lambda=kT_{xz}$
$R_{yt}-½g_{yt}R+g_{yt}\Lambda=kT_{yt}$	$R_{yx}-½g_{yx}R+g_{yx}\Lambda=kT_{yx}$	$R_{yy}-½g_{yy}R+g_{yy}\Lambda=kT_{yy}$	$R_{yz}-½g_{yz}R+g_{yz}\Lambda=kT_{yz}$
$R_{zt}-½g_{zt}R+g_{zt}\Lambda=kT_{zt}$	$R_{zx}-½g_{zx}R+g_{zx}\Lambda=kT_{zx}$	$R_{zy}-½g_{zy}R+g_{zy}\Lambda=kT_{zy}$	$R_{zz}-½g_{zz}R+g_{zz}\Lambda=kT_{zz}$

But only 10 of them (in black) are unique.

As we have seen, the Ricci scalar can be described in terms of the Ricci tensor. What we haven't mentioned is that the Ricci tensor can also be described in terms of just the metric (described in GR1d). So this equation is really a (very complicated) set of partial differential equations in $g_{\mu\upsilon}$. Given the stress-energy-momentum tensor, we can (in theory) solve for $g_{\mu\upsilon}$ and determine the curvature of space due to the given distribution of matter and energy.

Technically, it is a "coupled hyperbolic-elliptic non-linear 2nd order partial differential equation". Being coupled and non-linear (in multiple ways) makes it extremely hard to find an exact solution for $g_{\mu\upsilon}$.

Now for a neat trick – if we take Einstein's equation and multiply thru by the metric :

$$g^{\mu\upsilon}R_{\mu\upsilon} - ½g^{\mu\upsilon}g_{\mu\upsilon}\cdot R + \Lambda\cdot g^{\mu\upsilon}g_{\mu\upsilon} = k\cdot g^{\mu\upsilon}T_{\mu\upsilon}$$

we get (because $g^{\mu\upsilon}g_{\mu\upsilon} = \delta_\mu{}^\mu = 4$) : $R - ½\cdot 4R + 4\Lambda = -R + 4\Lambda = k\cdot T$ so $R = -k\cdot T + 4\Lambda$

Where $T = T_\mu{}^\mu$ is the trace of $T_{\mu\upsilon}$ (like R and $R_{\mu\upsilon}$). And if we substitute that back into the original equation, we eventually get

$$R_{\mu\upsilon} = k(T_{\mu\upsilon} - ½g_{\mu\upsilon}T) + \Lambda\cdot g_{\mu\upsilon}$$

If $\Lambda = 0$ (which is debatable), then what this is *essentially* saying is that the acceleration of particles (which is related to $R_{\mu\upsilon}$) near a given distribution of matter and energy (described by **T**) is proportional to **T** – T/2.

What Does it Mean?

J. A. Wheeler has described general relativity as "geometry tells matter how to move, and matter tells geometry how to curve".

Consider a small round ball of test particles that are all initially at rest relative to each other. If we start with such a ball of particles, it will, in the presence of curved spacetime, change size and shape as time passes. Let V(t) be the volume of the ball after a time t has elapsed (as measured by the particle at the center of the ball). Then, *assuming no Weyl curvature*, Einstein's equation says (remember, $R_{ab} \propto d^2V/dt^2$) :

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}\begin{pmatrix} \text{flow of } t\text{–momentum in } t \text{ direction} + \\ \text{flow of } x\text{–momentum in } x \text{ direction} + \\ \text{flow of } y\text{–momentum in } y \text{ direction} + \\ \text{flow of } z\text{–momentum in } z \text{ direction} \end{pmatrix}$$

where these flows are measured at the center of the ball at time zero, using local inertial coordinates. To keep things simple, these flows are just the *diagonal components* of the stress-energy-momentum tensor **T**. The flow of t-momentum in the t-direction is just the "energy density", ρ. The flow of x-momentum in the x-direction is the "pressure in the x direction" P_x, and similarly for y and z. We may then summarize Einstein's equation as :

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}(\rho + P_x + P_y + P_z)$$

This equation says that positive energy density and positive pressure will curve spacetime in a way that makes a freely falling ball of particles tend to shrink (due to the minus sign). Since $E = mc^2$, ordinary mass density counts as a form of energy density. Thus a massive object will make a swarm of freely falling particles initially at rest around it start to shrink. In short : the large object attracts other objects to it.

When the pressure is the same in every direction, Einstein's equation in plain English says : given a small ball of particles in free fall and initially at rest with respect to each other, the acceleration at which it shrinks is proportional to its volume times [the energy density plus three times the pressure] at the center of the ball.

It may seem odd that "pressure" causes gravitational attraction – in everyday life, pressure makes things want to expand (think of blowing up a balloon). Here, however, we are talking about gravitational effects of pressure, which has the same units as an energy density, and any energy causes spacetime to curve (see GR2f for a more in-depth explanation).

Is this all you need to understand general relativity? Of course not! All this was just the *bare minimum* needed to have some idea of what Einstein's equation is about. To be able to work problems in general relativity, you'd need to understand many other things in much more detail, including tetrads, Christoffel symbols, covariant derivatives, one-forms, Killing vectors, etc. That's what the GR2 and GR3 documents are all about.