Expanding Einstein's Equation

This section won't really help your understanding of Einstein's equation. It just shows how really complicated the equation is, by expanding out everything in terms of the metric tensor. The equation in its most common form is :

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\cdot R + g_{\mu\nu}\cdot\Lambda = k\cdot T_{\mu\nu}$$

But

$$R = g^{\alpha\beta}\cdot R_{\alpha\beta}$$

So

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\cdot\ g^{\alpha\beta}\cdot R_{\alpha\beta} + g_{\mu\nu}\cdot\Lambda = k\cdot T_{\mu\nu}$$

Now, since

$$R_{\mu\nu} = \partial(\Gamma^{\eta}{}_{\mu\nu})/\partial x^{\eta} - \partial(\Gamma^{\eta}{}_{\eta\mu})/\partial x^{\nu} + \Gamma^{\eta}{}_{\eta\lambda}\cdot\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\eta}\cdot\Gamma^{\eta}{}_{\nu\lambda}$$

(The Γ is a "Christoffel symbol" and is explained in detail in GR2c. Don't worry about it here.)

Then the equation becomes :

$$\partial(\Gamma^{\eta}{}_{\mu\nu})/\partial x^{\eta} - \partial(\Gamma^{\eta}{}_{\eta\mu})/\partial x^{\nu} + \Gamma^{\eta}{}_{\eta\lambda}\cdot\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\eta}\cdot\Gamma^{\eta}{}_{\nu\lambda}$$

$$- \tfrac{1}{2}\, g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial(\Gamma^{\eta}{}_{\alpha\beta})/\partial x^{\eta} + \tfrac{1}{2}\, g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial(\Gamma^{\eta}{}_{\eta\alpha})/\partial x^{\beta}$$

$$- \tfrac{1}{2}\, g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\Gamma^{\eta}{}_{\eta\lambda}\cdot\Gamma^{\lambda}{}_{\alpha\beta} + \tfrac{1}{2}\, g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\Gamma^{\lambda}{}_{\alpha\eta}\cdot\Gamma^{\eta}{}_{\beta\lambda}$$

$$+ g_{\mu\nu}\cdot\Lambda = k\cdot T_{\mu\nu}$$

But wait, it's about to get much worse, because the formula for the Christoffel symbol is :

$$\Gamma^{\gamma}{}_{\alpha\beta} = \tfrac{1}{2}\cdot g^{\delta\gamma}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} + \tfrac{1}{2}\cdot g^{\delta\gamma}\cdot\partial g_{\beta\delta}/\partial x^{\alpha} - \tfrac{1}{2}\cdot g^{\delta\gamma}\cdot\partial g_{\alpha\beta}/\partial x^{\delta}$$

So the equation becomes :

$$\tfrac{1}{2}\,\partial(g^{\delta\eta}\cdot\partial g_{\mu\delta}/\partial x^{\nu} + g^{\delta\eta}\cdot\partial g_{\nu\delta}/\partial x^{\mu} - g^{\delta\eta}\cdot\partial g_{\mu\nu}/\partial x^{\delta})/\partial x^{\eta}$$

$$-\tfrac{1}{2}\,\partial(g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\mu} + g^{\delta\eta}\cdot\partial g_{\mu\delta}/\partial x^{\eta} - g^{\delta\eta}\cdot\partial g_{\eta\mu}/\partial x^{\delta})/\partial x^{\nu}$$

$$+\tfrac{1}{4}\,(g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda} + g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta} - g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta})\cdot (g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\nu} + g^{\delta\lambda}\cdot\partial g_{\nu\delta}/\partial x^{\mu} - g^{\delta\lambda}\cdot\partial g_{\mu\nu}/\partial x^{\delta})$$

$$-\tfrac{1}{4}\,(g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\eta} + g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\mu} - g^{\delta\lambda}\cdot\partial g_{\mu\eta}/\partial x^{\delta})\cdot (g^{\delta\eta}\cdot\partial g_{\nu\delta}/\partial x^{\lambda} + g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\nu} - g^{\delta\eta}\cdot\partial g_{\nu\lambda}/\partial x^{\delta})$$

$$-\tfrac{1}{4}\,g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial(g^{\delta\eta}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} + g^{\delta\eta}\cdot\partial g_{\beta\delta}/\partial x^{\alpha} - g^{\delta\eta}\cdot\partial g_{\alpha\beta}/\partial x^{\delta})/\partial x^{\eta}$$

$$+\tfrac{1}{4}\,g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial(g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\alpha} + g^{\delta\eta}\cdot\partial g_{\alpha\delta}/\partial x^{\eta} - g^{\delta\eta}\cdot\partial g_{\eta\alpha}/\partial x^{\delta})/\partial x^{\beta}$$

$$-\tfrac{1}{8}\,g_{\mu\nu}\cdot g^{\alpha\beta}\cdot(g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda} + g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta} - g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta})\cdot (g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} + g^{\delta\lambda}\cdot\partial g_{\beta\delta}/\partial x^{\alpha} - g^{\delta\lambda}\cdot\partial g_{\alpha\beta}/\partial x^{\delta})$$

$$+\tfrac{1}{8}\,g_{\mu\nu}\cdot g^{\alpha\beta}\cdot(g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\eta} + g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\alpha} - g^{\delta\lambda}\cdot\partial g_{\alpha\eta}/\partial x^{\delta})\cdot (g^{\delta\eta}\cdot\partial g_{\beta\delta}/\partial x^{\lambda} + g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\beta} - g^{\delta\eta}\cdot\partial g_{\beta\lambda}/\partial x^{\delta})$$

$$+\, g_{\mu\nu}\cdot\Lambda = k\cdot T_{\mu\nu}$$

Now we can finally see that it is really “just” a (<u>very</u> complicated) 2nd-order partial differential equation in $g_{\mu\nu}$.

Expanding the differentials and multiplying out the ( ) · ( ) terms in the above, the equation becomes :

$$\tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\mu\delta}/\partial x^{\nu} + \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\mu\delta}/\partial x^{\nu}\partial x^{\eta}$$
$$+ \tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\nu\delta}/\partial x^{\mu} + \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\nu\delta}/\partial x^{\mu}\partial x^{\eta}$$
$$- \tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\mu\nu}/\partial x^{\delta} + \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\mu\nu}/\partial x^{\delta}\partial x^{\eta}$$

$$- \tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\nu}\cdot\partial g_{\eta\delta}/\partial x^{\mu} - \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\eta\delta}/\partial x^{\mu}\partial x^{\nu}$$
$$- \tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\nu}\cdot\partial g_{\mu\delta}/\partial x^{\eta} - \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\mu\delta}/\partial x^{\eta}\partial x^{\nu}$$
$$+ \tfrac{1}{2}\partial g^{\delta\eta}/\partial x^{\nu}\cdot\partial g_{\eta\mu}/\partial x^{\delta} - \tfrac{1}{2}g^{\delta\eta}\cdot\partial^{2}g_{\eta\mu}/\partial x^{\delta}\partial x^{\nu}$$

$$+ \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\nu} + \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\nu\delta}/\partial x^{\mu} - \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\nu}/\partial x^{\delta}$$
$$+ \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\nu} + \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\nu\delta}/\partial x^{\mu} - \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\nu}/\partial x^{\delta}$$
$$- \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\nu} - \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\nu\delta}/\partial x^{\mu} + \tfrac{1}{4}g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\mu\nu}/\partial x^{\delta}$$

$$- \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\nu\delta}/\partial x^{\lambda} - \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\nu} + \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\nu\lambda}/\partial x^{\delta}$$
$$- \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\mu}\cdot g^{\delta\eta}\cdot\partial g_{\nu\delta}/\partial x^{\lambda} - \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\mu}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\nu} + \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\mu}\cdot g^{\delta\eta}\cdot\partial g_{\nu\lambda}/\partial x^{\delta}$$
$$+ \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\nu\delta}/\partial x^{\lambda} + \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\nu} - \tfrac{1}{4}g^{\delta\lambda}\cdot\partial g_{\mu\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\nu\lambda}/\partial x^{\delta}$$

$$- \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} - \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\alpha\delta}/\partial x^{\beta}\partial x^{\eta}$$
$$- \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\beta\delta}/\partial x^{\alpha} - \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\beta\delta}/\partial x^{\alpha}\partial x^{\eta}$$
$$+ \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\eta}\cdot\partial g_{\alpha\beta}/\partial x^{\delta} - \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\alpha\beta}/\partial x^{\delta}\partial x^{\eta}$$

$$+ \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\beta}\cdot\partial g_{\eta\delta}/\partial x^{\alpha} + \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\eta\delta}/\partial x^{\alpha}\partial x^{\beta}$$
$$+ \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\beta}\cdot\partial g_{\alpha\delta}/\partial x^{\eta} + \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\alpha\delta}/\partial x^{\eta}\partial x^{\beta}$$
$$- \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot\partial g^{\delta\eta}/\partial x^{\beta}\cdot\partial g_{\eta\alpha}/\partial x^{\delta} + \tfrac{1}{4}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial^{2}g_{\eta\alpha}/\partial x^{\delta}\partial x^{\beta}$$

$$- \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} - \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\beta\delta}/\partial x^{\alpha}$$
$$+ \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\delta}/\partial x^{\lambda}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\beta}/\partial x^{\delta} - \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\beta}$$
$$- \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\beta\delta}/\partial x^{\alpha} + \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\eta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\beta}/\partial x^{\delta}$$
$$+ \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\beta} + \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\beta\delta}/\partial x^{\alpha}$$
$$- \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\eta}\cdot\partial g_{\eta\lambda}/\partial x^{\delta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\beta}/\partial x^{\delta}$$

$$+ \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\beta\delta}/\partial x^{\lambda} + \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\beta}$$
$$- \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\delta}/\partial x^{\eta}\cdot g^{\delta\eta}\cdot\partial g_{\beta\lambda}/\partial x^{\delta} + \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\alpha}\cdot g^{\delta\eta}\cdot\partial g_{\beta\delta}/\partial x^{\lambda}$$
$$+ \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\alpha}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\beta} - \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\eta\delta}/\partial x^{\alpha}\cdot g^{\delta\eta}\cdot\partial g_{\beta\lambda}/\partial x^{\delta}$$
$$- \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\beta\delta}/\partial x^{\lambda} - \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\lambda\delta}/\partial x^{\beta}$$
$$+ \tfrac{1}{8}g_{\mu\nu}\cdot g^{\alpha\beta}\cdot g^{\delta\lambda}\cdot\partial g_{\alpha\eta}/\partial x^{\delta}\cdot g^{\delta\eta}\cdot\partial g_{\beta\lambda}/\partial x^{\delta}$$

$$+ g_{\mu\nu}\cdot\Lambda = k\cdot T_{\mu\nu}$$

Finally, by showing all the summations that are implied by the Einstein notation, where $\sum_\delta$ means to sum over $\delta=0,1,2,3$, the equation becomes :

$$\begin{aligned}
\textstyle\sum_\delta \sum_\eta \{ \tfrac{1}{2} (&\partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\mu\delta}/\partial x^\nu + g^{\delta\eta} \cdot \partial^2 g_{\mu\delta}/\partial x^\nu \partial x^\eta \\
&+ \partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\nu\delta}/\partial x^\mu + g^{\delta\eta} \cdot \partial^2 g_{\nu\delta}/\partial x^\mu \partial x^\eta \\
&- \partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\mu\nu}/\partial x^\delta + g^{\delta\eta} \cdot \partial^2 g_{\mu\nu}/\partial x^\delta \partial x^\eta)
\end{aligned}$$

this would be <u>96 terms</u> if written out completely!

$$\begin{aligned}
- \tfrac{1}{2} (&\partial g^{\delta\eta}/\partial x^\nu \cdot \partial g_{\eta\delta}/\partial x^\mu + g^{\delta\eta} \cdot \partial^2 g_{\eta\delta}/\partial x^\mu \partial x^\nu \\
&+ \partial g^{\delta\eta}/\partial x^\nu \cdot \partial g_{\mu\delta}/\partial x^\eta + g^{\delta\eta} \cdot \partial^2 g_{\mu\delta}/\partial x^\eta \partial x^\nu \\
&- \partial g^{\delta\eta}/\partial x^\nu \cdot \partial g_{\eta\mu}/\partial x^\delta + g^{\delta\eta} \cdot \partial^2 g_{\eta\mu}/\partial x^\delta \partial x^\nu) \qquad \text{96 terms!}
\end{aligned}$$

$$\begin{aligned}
+ \textstyle\sum_\alpha \sum_\beta [-\tfrac{1}{4} g_{\mu\nu} \cdot (&g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\alpha\delta}/\partial x^\beta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\alpha\delta}/\partial x^\beta \partial x^\eta \\
&+ g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\beta\delta}/\partial x^\alpha + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\beta\delta}/\partial x^\alpha \partial x^\eta \\
&- g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\eta \cdot \partial g_{\alpha\beta}/\partial x^\delta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\alpha\beta}/\partial x^\delta \partial x^\eta) \qquad \text{1536 terms!}
\end{aligned}$$

$$\begin{aligned}
+ \tfrac{1}{4} g_{\mu\nu} \cdot (&g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\beta \cdot \partial g_{\eta\delta}/\partial x^\alpha + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\eta\delta}/\partial x^\alpha \partial x^\beta \\
&+ g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\beta \cdot \partial g_{\alpha\delta}/\partial x^\eta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\alpha\delta}/\partial x^\eta \partial x^\beta \\
&- g^{\alpha\beta} \cdot \partial g^{\delta\eta}/\partial x^\beta \cdot \partial g_{\eta\alpha}/\partial x^\delta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial^2 g_{\eta\alpha}/\partial x^\delta \partial x^\beta) \qquad \text{1536 terms!}
\end{aligned}$$

$$\begin{aligned}
+ \textstyle\sum_\lambda \Big(- \tfrac{1}{8} g_{\mu\nu} \cdot (&g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\beta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\beta\delta}/\partial x^\alpha \\
&- g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\beta}/\partial x^\delta + g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\beta \\
&+ g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\beta\delta}/\partial x^\alpha - g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\beta}/\partial x^\delta \\
&- g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\beta - g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\beta\delta}/\partial x^\alpha \\
&+ g^{\alpha\beta} \cdot g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\beta}/\partial x^\delta) \qquad \text{9216 terms!}
\end{aligned}$$

$$\begin{aligned}
+ \tfrac{1}{8} g_{\mu\nu} \cdot (&g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\eta \cdot g^{\delta\eta} \cdot \partial g_{\beta\delta}/\partial x^\lambda + g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\eta \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\beta \\
&- g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\delta}/\partial x^\eta \cdot g^{\delta\eta} \cdot \partial g_{\beta\lambda}/\partial x^\delta + g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^\alpha \cdot g^{\delta\eta} \cdot \partial g_{\beta\delta}/\partial x^\lambda \\
&+ g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^\alpha \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\beta - g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^\alpha \cdot g^{\delta\eta} \cdot \partial g_{\beta\lambda}/\partial x^\delta \\
&- g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\eta}/\partial x^\delta \cdot g^{\delta\eta} \cdot \partial g_{\beta\delta}/\partial x^\lambda - g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\eta}/\partial x^\delta \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\beta \\
&+ g^{\alpha\beta} \cdot g^{\delta\lambda} \cdot \partial g_{\alpha\eta}/\partial x^\delta \cdot g^{\delta\eta} \cdot \partial g_{\beta\lambda}/\partial x^\delta) \qquad \text{9216 terms!}
\end{aligned}$$

$$\begin{aligned}
+ \tfrac{1}{4} (&g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^\nu + g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\nu\delta}/\partial x^\mu \\
&- g^{\delta\eta} \cdot \partial g_{\eta\delta}/\partial x^\lambda \cdot g^{\delta\lambda} \cdot \partial g_{\mu\nu}/\partial x^\delta + g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^\nu \\
&+ g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\nu\delta}/\partial x^\mu - g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^\eta \cdot g^{\delta\lambda} \cdot \partial g_{\mu\nu}/\partial x^\delta \\
&- g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^\nu - g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\nu\delta}/\partial x^\mu \\
&+ g^{\delta\eta} \cdot \partial g_{\eta\lambda}/\partial x^\delta \cdot g^{\delta\lambda} \cdot \partial g_{\mu\nu}/\partial x^\delta) \qquad \text{576 terms!}
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{4} \, (g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^{\eta} \cdot g^{\delta\eta} \cdot \partial g_{\nu\delta}/\partial x^{\lambda} + g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^{\eta} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^{\nu} \\
&\quad - g^{\delta\lambda} \cdot \partial g_{\mu\delta}/\partial x^{\eta} \cdot g^{\delta\eta} \cdot \partial g_{\nu\lambda}/\partial x^{\delta} + g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^{\mu} \cdot g^{\delta\eta} \cdot \partial g_{\nu\delta}/\partial x^{\lambda} \\
&\quad + g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^{\mu} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^{\nu} - g^{\delta\lambda} \cdot \partial g_{\eta\delta}/\partial x^{\mu} \cdot g^{\delta\eta} \cdot \partial g_{\nu\lambda}/\partial x^{\delta} \\
&\quad - g^{\delta\lambda} \cdot \partial g_{\mu\eta}/\partial x^{\delta} \cdot g^{\delta\eta} \cdot \partial g_{\nu\delta}/\partial x^{\lambda} - g^{\delta\lambda} \cdot \partial g_{\mu\eta}/\partial x^{\delta} \cdot g^{\delta\eta} \cdot \partial g_{\lambda\delta}/\partial x^{\nu} \\
&\quad + g^{\delta\lambda} \cdot \partial g_{\mu\eta}/\partial x^{\delta} \cdot g^{\delta\eta} \cdot \partial g_{\nu\lambda}/\partial x^{\delta}) \qquad \text{576 terms!}
\end{aligned}
$$

$$ \} \quad] \quad \Big) $$

$$ + g_{\mu\nu} \cdot \Lambda = k \cdot T_{\mu\nu} $$

And no, I am not going to re-write it one more time with all 22,848 terms!

As if the above equation (yes, remember it's all one equation!) isn't complicated enough :

- repeat this equation 9 times, for the various values of μ and ν = (t,x,y,z)!
- each "$g_{\mu\nu} \cdot g^{\alpha\beta}$" (or similar) term is a *matrix multiplication*, requiring many individual multiplications and additions (these terms are also one of the things which makes the equation non-linear)
- $g_{\alpha\beta}$ is the matrix inverse of $g^{\alpha\beta}$, which not only requires many multiplications and additions to calculate, but makes the differential equation even more non-linear!
- if you choose a different coordinate system (such as spherical), the format of the derivatives become even more complicated!

Solving the Equation

In general, Einstein's equation by itself does not contain enough information to determine $g_{\mu\nu}$. Various additional equations, including equations of state, constraint equations, continuity equations, and/or gauge conditions are needed to provide enough information to uniquely determine $g_{\mu\nu}$. These tend to be problem-specific, and are too complicated to describe here simply, but are discussed in GR3x.

Once you find $g_{\mu\nu}$, you calculate the line element, which tells you how long a small piece of stretched spacetime actually is :

$$ ds^2 = g_{\mu\nu} \cdot dx^u \cdot dx^v $$

$$ = \textstyle\sum_{\mu}\sum_{\nu} [g_{\mu\nu} \cdot dx^u \cdot dx^v] $$

Which in Cartesian coordinates (for example) becomes

$$
\begin{aligned}
ds^2 = \; & g_{00}dt \cdot dt + g_{01}dt \cdot dx + g_{02}dt \cdot dy + g_{03}dt \cdot dz \\
& + g_{10}dx \cdot dt + g_{11}dx \cdot dx + g_{12}dx \cdot dy + g_{13}dx \cdot dz \\
& + g_{20}dy \cdot dt + g_{21}dy \cdot dx + g_{22}dy \cdot dy + g_{23}dy \cdot dz \\
& + g_{30}dz \cdot dt + g_{31}dz \cdot dx + g_{32}dz \cdot dy + g_{33}dz \cdot dz
\end{aligned}
$$

Then you can compute the paths of objects in spacetime by using the *geodesic equation*, which gives four *equations of motion*. Various initial conditions and conservation laws may also be needed. The geodesic equation for **no non-gravitational forces** is :

$$\partial^2 x^\alpha/\partial\tau^2 + \Gamma^\alpha{}_{\beta\gamma}\cdot(\partial x^\beta/\partial\tau)\cdot(\partial x^\gamma/\partial\tau) = 0$$

this gives 4 equations, for $\alpha = (t,x,y,z)$

or

$$\partial^2 x^\alpha/\partial\tau^2 + ½(g^{\delta\alpha}\cdot\partial g_{\beta\delta}/\partial x^\gamma + g^{\delta\alpha}\cdot\partial g_{\gamma\delta}/\partial x^\beta - g^{\delta\alpha}\cdot\partial g_{\beta\gamma}/\partial x^\delta)\cdot(\partial x^\beta/\partial\tau)\cdot(\partial x^\gamma/\partial\tau) = 0$$

which becomes

$$\partial^2 x^\alpha/\partial\tau^2 + ½\sum_\beta\sum_\gamma\sum_\delta [g^{\delta\alpha}\cdot(\partial g_{\beta\delta}/\partial x^\gamma + \partial g_{\gamma\delta}/\partial x^\beta - \partial g_{\beta\gamma}/\partial x^\delta)\cdot(\partial x^\beta/\partial\tau)\cdot(\partial x^\gamma/\partial\tau)] = 0$$

And then you're done! Simple, huh? ☺