These GR2x files go deeper into the mathematics and terminology needed to understand papers on general relativity and such. This file describes more of the terms and operators used with tensors, and many miscellaneous general definitions.

More on Vectors and Tensors

Tensors (in general) not only have a rank, but a type. A **tensor of type (m,n)** means several things. First, it has *m* contravariant/upper indices and *n* covariant/lower indices. The rank is just (m+n), which tells you how many indexes it has. For example : a scalar is type (0,0) and has no rank (single number), an ordinary (contravariant) spacetime vector V^i is type (1,0) which has rank 1 (with 4 dimensions). The metric tensor g_{uv} is type (0,2) and is rank 2 with 4x4 dimensions.

Another way to look at a tensor is as an operator or function that takes in vectors and returns a result (as was mentioned in GR1a, b, and c). If you think of a tensor like this, it takes in *m* covariant vectors and *n* contravariant vectors as arguments (because of the Einstein summation convention, you have to "feed" a covariant index with a contravariant vector, and vise-versa) and it returns a scalar. For example : The metric tensor g_{uv} is type (0,2) because it takes 2 contravariant vectors and returns a scalar (their "dot product"). The inverse metric tensor g^{uv} is type (2,0) and takes 2 covariant vectors and likewise returns a scalar.

But, we do not have to give a tensor all the vectors it needs! Then the result is a tensor of type (a,b), where

a = m – #input covariant vectors　　　　b = n – #input contravariant vectors

For example, the Riemann tensor $R^a{}_{bcd}$ is a (1,3) tensor that takes three contravariant vectors as inputs, and outputs one covariant vector. In the diffusion formula from GR1a : $J_i = -D_i^j C_{,j}$ and D is (1,1) while $C_{,j}$ is (1,0) so J_i is (0,1). Thus **J** can have both a different length and direction than **C**.

Tensors can also be **contracted** by summing over one upper and one lower index, turning a type (m,n) tensor into a type (m-1,n-1) tensor, and is often called the **trace**. The Ricci scalar $R = R^\lambda{}_\lambda$ is an example of this.

A tensor is **symmetric** if it is the same when two indexes are swapped : $T_{\alpha\beta} = T_{\beta\alpha} = T_{(\alpha\beta)}$ and **anti-symmetric** or **skew-symmetric** if it changes sign : $T_{\alpha\beta} = -T_{\beta\alpha} = T_{[\alpha\beta]}$

Any tensor T_{ab} can be split into **symmetric** () and **anti-symmetric** [] parts :

$$T_{(ab)} = ½ (T_{ab} + T_{ba})$$
$$T_{[ab]} = ½ (T_{ab} - T_{ba})$$

The metric tensor is symmetric : $g_{[ab]} = 0$

There is another kind of bracket notation called "cyclic notation", which means that the indexes should be rotated (with wrap-around, not changing the initial relationships) to create multiple terms with alternating signs :

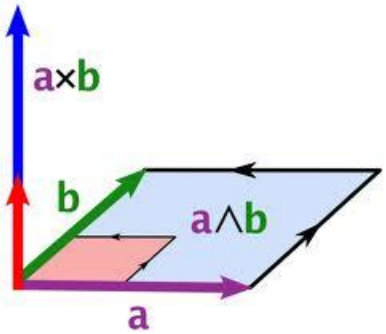
$$R_{a[bcd]} \rightarrow R_{abcd} - R_{adbc} + R_{acdb}$$
$$F_{[ab,c]} \rightarrow F_{ab,c} - F_{ca,b} + F_{bc,a} \rightarrow \partial_c F_{ab} - \partial_b F_{ca} + \partial_a F_{bc}$$
$$\nabla_{[\mu} V_{\nu]} \rightarrow \nabla_\mu V_\nu - \nabla_\nu V_\mu$$

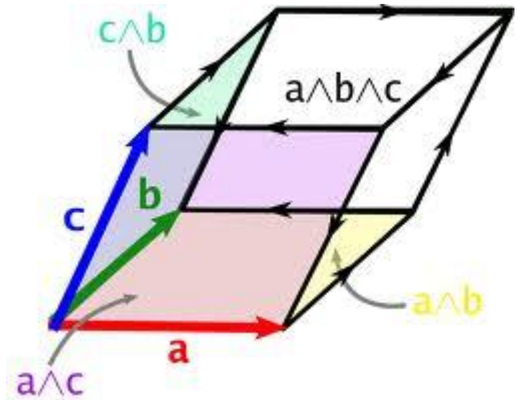
The **d'Alembertian operator**, $\Box^2$, is the spacetime equivalent of ∇^2 for general relativity :

$$\Box^2 = \nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu = (\partial^2/\partial t^2)/c^2 - \partial^2/\partial x^2 - \partial^2/\partial y^2 - \partial^2/\partial z^2$$

And returns a Lorentz-invariant value. Note : some authors use just $\Box$ to mean the exact same thing!

The “**exterior product**” or “**wedge product**” ∧ of two 3-D vectors is related to their cross product and to the determinant of the matrix formed by them. Also, given N N-dimensional vectors, it is related to the area/volume/etc. of the parallelogram defined by them.





The result of a∧b is called a **bivector**, and can be interpreted as an oriented plane segment, much as vectors can be thought of as directed line segments. The wedge product has the following properties :

$\mathbf{a} \wedge \mathbf{a} = 0$

$\mathbf{a} \wedge \mathbf{b} = -(\mathbf{b} \wedge \mathbf{a})$

$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a}\wedge\mathbf{b} + \mathbf{a}\wedge\mathbf{c}$

The “**dyadic product**”, “**tensor product**”, or “**outer product**” ⊗ of two vectors creates a tensor. In matrix format, it is the result of multiplying the column vector **u** by the row vector **v** :

$$\mathbf{u} \otimes \mathbf{v} \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

⊗ <u>is only used with vectors</u>; in index notation it disappears : $u^i v^j$ (note that there is no summation!).

The type of the result depends on the type of the vectors : $P_{ij} = u_i v_j$ and $P^i{}_j = u^i v_j$ (still no summation!).

Taking the dyadic product of a type (a,b) tensor with a type (c,d) tensor creates a type (a+c,b+d) tensor.

So any time you multiply the elements of vectors together <u>without summation</u>, you are making a dyadic product! Many authors do not show this symbol when multiplying vectors or basis terms, but for example strictly speaking the metric tensor is

$\mathbf{g} = g_{uv}(\mathbf{e}^u \otimes \mathbf{e}^v)$ $\qquad\qquad$ $\mathbf{g}^{-1} = g^{uv}(\mathbf{e}_u \otimes \mathbf{e}_v)$

So that $(\mathbf{e}^u \otimes \mathbf{e}^v)$ creates the 4x4 “structure” of the tensor, which is then filled in by the terms g_{uv}, in the same way that a vector is built from its components by individual basis terms : $\mathbf{V} = V^a \mathbf{e}_a$ but we usually just write V^a.

While rank-2 tensors of any type are often shown in **matrix format**, it is only the “mixed” tensor $T^u{}_v$ that is accurately represented by a matrix. This is because a matrix can <u>only</u> be formed from the dyadic product of a column vector **u** (4x1) and a row vector **v** (1x4) : $\mathbf{u} \otimes \mathbf{v}^T = u^i v_j \rightarrow T^i{}_j$. While other types of rank-2 tensors can be *displayed* in a matrix format, they do not follow the rules of matrix math. And because $g^{ui} g_{iv} = \delta^u{}_v$, tensors in $T^u{}_v$ format don’t have any terms from the coordinate system in them.

The **Levi-Civita symbol ε** (in N dimensions) is a tensor with N indexes such that it equals +1 if the indexes are an even permutation (ascending order, with wraparound, like 1,2,3… or 3,4,…,1,2) and –1 if they are an odd permutation (descending order), and 0 otherwise. Some authors use η instead.

Note that $\varepsilon^{ijk} = \varepsilon_{ijk}$ and $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$ and $\varepsilon_{ijk} = \varepsilon_{jik} = \varepsilon_{kij}$

In 2-dimensions:

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In 3-dimensions:

$$\epsilon^{ab1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\epsilon^{ab2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\epsilon^{ab3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For example, ε is used to create cross-products in index notation : $(\nabla \times \mathbf{A})^i = \varepsilon^{ijk}\partial_j A_k$

The y-component of $\nabla \times \vec{\mathbf{A}}$ is

$$\begin{aligned}
\left(\nabla \times \vec{\mathbf{A}}\right)_y &= \epsilon^{2jk}\partial_j A_k \qquad \text{(2nd component = y-component)} \\
&= \epsilon^{231}\partial_3 A_1 + \epsilon^{213}\partial_1 A_3 \\
&= (1)\frac{\partial}{\partial z}A_x + (-1)\frac{\partial}{\partial x}A_z \\
&= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}.
\end{aligned}$$

Definition 3.7. The *covariant and contravariant scalar products* of two rank 2 tensors, S and T, are defined as $g^{ik}g^{jl}S_{kl}T_{ij}$ and $g_{ik}g_{jl}S^{ij}T^{kl}$ respectively.

Similar to the scalar product of two rank 1 tensors, these operations result in a scalar:

$$g^{ik}g^{jl}S_{kl}T_{ij} = S^{ij}T_{ij} = S_{kl}g^{ik}g^{jl}T_{ij} = S_{kl}T^{kl} = g_{ik}g_{jl}S^{ij}T^{kl}.$$

Thus, the covariant and contravariant scalar products of two rank 2 tensors give the same value and are collectively referred to simply as the *scalar product*.

More Terms and Definitions

A **linear** function f(x) satisfies the following properties :

f(x + y) = f(x) + f(y)
f(a·x) = a·f(x)

A "**de-Sitter**" universe is one in which the curvature $R > 0$ (sphere); an "**anti de-Sitter**" universe has curvature $R < 0$ (saddle).

The "**no-hair**" theorem states that stationary, asymptotically flat black hole solutions to general relativity can be fully described by only their mass, electric charge, and angular momentum.

A **holonomic basis** (or "holonomic coordinates", or a "coordinate basis") is one :

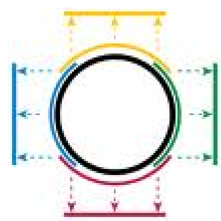
- which has coordinates x^a such that the bases are $\mathbf{e}_a = \partial/\partial x^a$
- whose metric tensor is g^{uv}
- that represents the global spacetime
- is "in" the surface

Otherwise, a "**non(-)holonomic**", "anholonomic", or "non-coordinate basis", "orthonormal basis", "tetrad", "orthonormal tetrad" is one :

- whose metric tensor is the Minkowski metric
- in which the tangent basis vectors are orthonormal
- that represents a local Lorentz frame ("rest-frame", "proper")
- is "in" the tangent plane

Years ago, holonomic bases were thought to be essential to general relativity, so understanding the difference between contravariant and covariant was important. More recently, it seems that non-holonomic bases are actually easier to use! The formulas for the components of vectors and tensors are generally simpler, have a physical interpretation that's easier to understand, and allows general relativity to relate a little better to quantum mechanics. Cartesian, polar, and spherical coordinates in Euclidean space are holonomic. But the spherical basis $[\mathbf{e}_r, \mathbf{e}_\theta/r, \mathbf{e}_\varphi/(r\sin\theta)]$ is non-holonomic and orthonormal, *so there is no difference between the contravariant and covariant components*. Some authors use Greek and Roman indexes to differentiate between holonomic and non-holonomic bases instead of between [ct,x,y,z] and [x,y,z].

A **coordinate chart** is a non-singular coordinate system that spans all or part of a surface. Points on the surface are then projected onto the chart(s). A surface may have multiple coordinate charts if parts of one chart become ill-defined (like at the north and south poles of a spherical coordinate system). In addition, each *reference frame* may have its own coordinate chart, in which case the chart is chosen so that locally the coordinates are a flat Minkowski spacetime. The figure to the right shows four charts that completely cover the curved black surface. An **atlas** is a collection of charts.

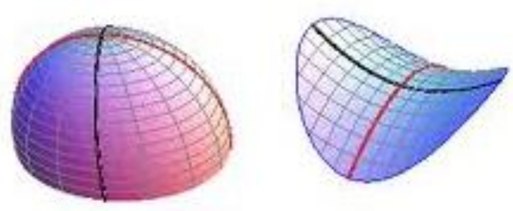


An **ansatz** is the assumed *general format* of the solution(s) to a problem. After an ansatz has been described, the equations are solved to find the result and *see if it works*. For example, a set of data may look like it is clustered about a straight line, so a linear ansatz y=ax+b could be used to find the best value of the parameters *a* and *b*, and then the quality of the fit determines whether the assumption was valid.

Two metrics g^{ab} and $\hat{g}^{ab}$ are **conformal** if $g^{ab} = \lambda^2 \hat{g}^{ab}$ for some non-zero differentiable function $\lambda(x^a)$.

Intrinsic vs. Extrinsic Curvature

In GR1a and GR1e it was mentioned that the usual pictures of curved 2-D surfaces → are not a perfect analogy for curved spacetime. The curvature that's visible in such an image is curving "outside" the surface into the 3-D space it is embedded in within, which is not the case with spacetime. In GR1e, flat 2-D surfaces were presented in which the space is locally curved while staying globally flat, which is



a better image. This is the difference between internal (or intrinsic) and external (or extrinsic) curvature.

In addition, it was mentioned that a creature living in these surfaces could not tell “just by looking” that its space was curved. It is very important to keep in mind that this creature is in the surface, not on the surface (like a bacteria in the film of a soap bubble, not a bug walking on a balloon).

Extrinsic curvature is the curvature seen from “outside” the space, like when we embed a 2-D surface within a 3-D space; intrinsic curvature is measured by the Riemann tensor in the surface and is a property of the surface, not how it is embedded in a higher-dimensional space.

Intrinsic vs. extrinsic curvature can be very counter-intuitive : for example, 1-D lines (like circles) have no intrinsic curvature, because there is no loop along which we can parallel transport (which is what the Riemann tensor does). The belief that a circle is curved comes from thinking of it when it is embedded in a flat 2-D plane, so its curvature is extrinsic. Imagine a 1-D creature living in a line : all it knows is that it can move forward or backward; there is *no way* for it to tell whether its universe (the line) is straight or curved.

The 2-D surface of a sphere has the same distance to the 3-D center (radius) at every point. More general curved 2-D surfaces have different radii at different points (like an ellipsoid). But a radius is something that we can only talk about from an external point of view. We can see that the sphere is curved because it is embedded within a 3-D space in which we are visualizing it. Internally, we can only talk about the results of a parallel transport around a tiny parallelogram that is in the surface. But there are ways that a creature living within the surface can notice that its space is not flat, such as determining that the sum of the angles in a triangle is more than 180º.

However, even “seeing” the curvature of a 2-D surface in a 3-D space can be misleading : if the creature lived in a 2-D flat surface (plane), the sum of the angles in a triangle would be exactly 180º. If this sheet is then rolled up into a cylinder, *the angles of a triangle still add up to 180º*. So the surface of a cylinder is not intrinsically curved, it just *looks* curved to us because of how it is embedded in the 3-D space we see it in. From an intrinsic point of view, a flat surface and a cylinder are no different, because if we started with a cylinder, we could unroll it and end up with a flat sheet. But if we start with the surface of a sphere, we cannot unroll it to a flat sheet (without tearing it), so it has a different intrinsic curvature.

The point is, since we cannot visualize 4-D spacetimes, it is very useful to present pictures of curved 2-D surfaces embedded within a 3-D space, but these visual analogies can be misleading. When we explore our 4-D spacetime we are like the 2-D creature : we can only look at spacetime and measure it from the inside, so we can only talk about its internal curvature. And we cannot assume the existence of any higher dimensional space in which it is embedded.

Flat Spacetimes

There are actually several different definitions of what it means for a spacetime to be “flat”.

If there is a coordinate system for which the metric components $g^{\mu\nu}$ are *constant* everywhere, then $\partial g^{\mu\nu}/\partial x^i = 0$, so $R^{\alpha}{}_{\beta\gamma\delta} = 0$ and the surface is globally flat. This is equivalent to saying that there exists a coordinate system in which the metric is equal to the Minkowski metric $\eta^{ab} = \text{diag}(1,-1,-1,-1)$.

A solution to Einstein’s equation is **asymptotically flat** if the metric approaches η^{ab} far from the origin, such as the Schwarzschild and Kerr solutions.

A solution is **conformally flat** if $g^{ab} = \lambda^2(x^a)\ \eta^{ab}$, which also means the Weyl tensor is zero. For example, in a constant, uniform gravitational field with acceleration *a* in the *z* direction we get the Rindler metric :

$$ds^2 = (1+az)^2dt^2 - dx^2 - dy^2 - dz^2$$

It can be shown with a transformation to the local frame of reference for a free-falling, accelerating observer that this metric is a multiple of η^{ab}.

Locally flat means that a "small enough" region around a point looks like Minkowski spacetime, at least for an instant. This means that $g_{uv} \approx \eta_{uv}$ and $\partial g_{uv}/\partial x^i = 0$, so that all first-order effects of gravity vanish, but not second-order effects (such as tides).

A spacetime is **Ricci flat** if the Ricci tensor and Λ are zero (no sources of mass-energy within the region being considered = "vacuum solution"), but the Weyl curvature does not have to be zero. The Schwarzschild and Kerr solutions are Ricci flat (because they describe the spacetime *outside* the object).