It couldn't hurt to first review GR1b for the basic background on special relativity! Reminders :

$\beta = v / c$ is always ≤ 1

$\gamma = 1 / (1-\beta^2)^{1/2}$ is always ≥ 1

The **magnitude** or **norm** of a vector **V** is

$$\mathbf{V}\cdot\mathbf{V} = ds^2 = \eta_{uv}V^uV^v = c^2dt^2 - dx^2 - dy^2 - dz^2$$ (in Cartesian coordinates, for example)

Forms of the Minkowski Metric

In GR1b, the metric tensor was defined as

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Which for simplicity will be described as "diag(1,–1,–1,–1)", and referred to as the **metric signature**. Sometimes the metric signature is the sum of the diagonal, or –2 here. This means the coordinate system is [ct, x, y, z]. Note that *ct* has units of meters if we pick the basic unit of time to be $3.34\text{x}10^{-9}$ seconds. Many authors use diag(–1,1,1,1); in both of these notations, *c*=1 (so it can be left out of equations for brevity) and the value of a velocity *v* is its fraction of the speed of light. Others use diag(c,–1,–1,–1) or diag(–c,1,1,1) so you can see where the *c* terms are for clarity, in which case the value of a velocity is in meters/second. Some even put the time term at the end : diag(–1,–1,–1,1) or diag(1,1,1,–1) or diag(–1,–1,–1,c) or diag(1,1,1,–c).

For the rest of this document, c will be explicitly shown for clarity and [ct,x,y,z] is used instead of $[x^0,x^1,x^2,x^3]$.

Relationship Between t and τ

Consider a particle moving with a constant velocity *v* in S (the "observer's" frame), and define S' so that its origin is always on the particle (the particle's "rest frame").

Now consider a small change in time at particle in S' :	$ds^2 = c^2dt'^2$
But in S this will be :	$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
Since ds is invariant :	$c^2dt'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
Multiply and divide part of the right side by dt^2 :	$c^2dt'^2 = c^2dt^2 - (dx^2/dt^2 + dy^2/dt^2 + dz^2/dt^2)dt^2$
So :	$c^2dt'^2 = (c^2 - v^2)dt^2$
Divide both sides by c^2 :	$dt'^2 = (1 - v^2/c^2)dt^2$
Since by definition dt' = τ :	$d\tau = (1 - v^2/c^2)^{1/2}\ dt$
And by the definition of γ :	$d\tau = dt / \gamma$

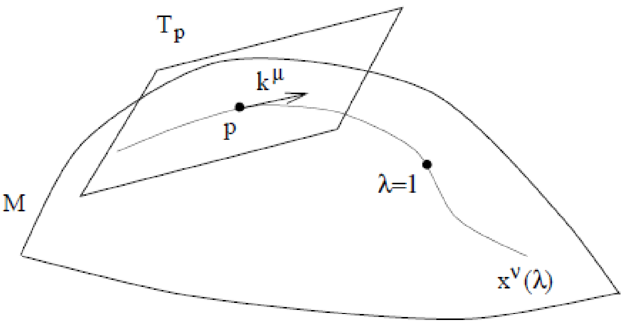
So dt (= γ dτ) will always be bigger than dτ, and t (= γτ) will always be bigger than τ.

Timelike, Null, Spacelike

A vector V^u with a positive norm is called **timelike**. If the norm is zero (which means its time-length *equals* its space-length!), the vector is **null** or **lightlike**, and if it's negative, the vector is **spacelike**. Note that these definitions are reversed for metric signatures whose sum is +2.

Particles travel along *worldlines* thru spacetime. In ordinary 3-D space, we might define the (x,y,z) positions of a path or curve as a function of time. But in spacetime, time is one of the dimensions! So instead we describe all four (t,x,y,z) positions as a function of a different parameter, λ. → For each value of λ, the values $x^\nu(\lambda)$, where "x" represents a position, not an axis, define a point on the curve. **$x^\nu(\lambda)$ is a shorthand for $x^\nu=f^\nu(\lambda)$**, or

$x=x^1=f^1(\lambda)$ $y=x^2=f^2(\lambda)$ $z=x^3=f^3(\lambda)$



A **path tangent vector** (k^μ) to the curve can then be calculated :

$k^\mu = (dx^\nu/d\lambda)\mathbf{e}_\nu$ (again, "x" represents position as a function of λ)

When the worldline is that of a particle with mass (timelike), the parameter represents the time passed in the reference frame that is travelling along the curve, which is the proper time τ. The worldline of a photon can't be parameterized by proper time, since for it $ds^2 = 0$ and so $d\tau = 0$, but we can use other parameters such as the coordinate time t in some reference frame. Some authors use τ for timelike, *s* for spacelike, and λ for null as parameters. Others just always use *s* or always use λ. Here λ is always used.

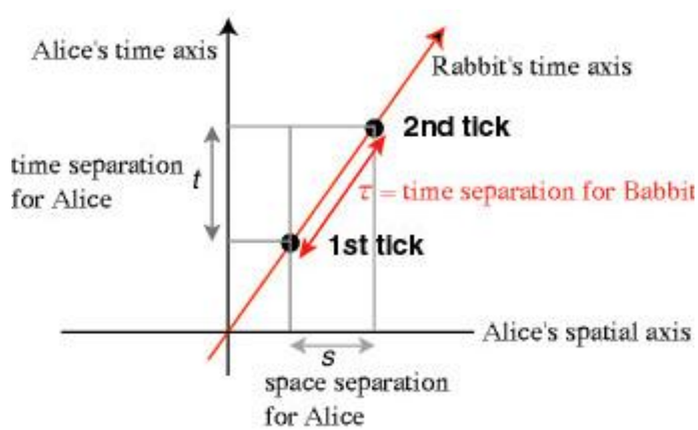
A worldline (or "curve", or "path") $x^a(\lambda)$ is timelike, null, or spacelike if its path tangent vector $dx^a/d\lambda$ is timelike, null, or spacelike at every point on the curve.

All ordinary objects follow timelike worldlines. Light travels on "null geodesics" – on geodesics with a norm of zero. If a path between two events is spacelike, it would require moving faster than light to get between the starting and ending events.

A worldline that is everywhere either timelike or null is sometimes called a **causal curve**, as it is a path that a message can travel along.

Spacetime Diagrams

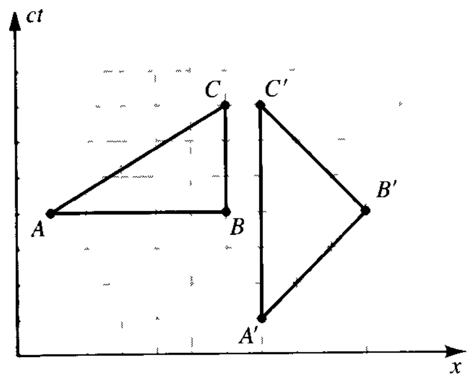
Spacetime diagrams traditionally show one space dimension horizontally and the time dimension vertically, thus showing the close relationship between space and time. In a 2-D spacetime diagram, the interval (distance) between two points is $ds^2 = dt^2 – dx^2$. The fact that the spatial terms are being subtracted from the time term results in some counter-intuitive results about "distance" in spacetime diagrams. For example, the figure to the right represents the frame of Alice, who sees a rabbit moving away from her carrying a watch. If we consider two ticks of the rabbit's watch to be "events", the figure shows that between tick one and two the rabbit has moved a certain distance away from her during a certain time interval. The problem is, Euclidean geometry says that for Alice the distance between the two events is $(t^2 + s^2)^{1/2} = \tau$ when it is really $(t^2 – s^2)^{1/2}$, a much smaller value!



In the left-hand triangle in the following figure, the lengths of the sides are :

$AB^2 = 0^2 – 5^2 = –25$ $BC^2 = 3^2 – 0^2 = 9$ $CA^2 = 3^2 – 5^2 = –14$

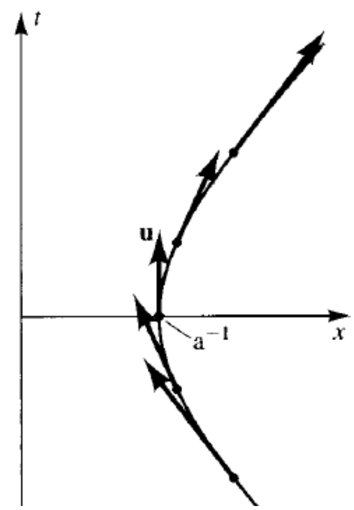
So in spacetime (ignoring negatives), this hypotenuse is *shorter* than the horizontal! The shortest spacetime interval from A to C is AC.



In the right-hand triangle, the lengths of the sides are :

$A'B'^2 = 3^2 - 3^2 = 0$ $B'C'^2 = 3^2 - 3^2 = 0$ $C'A'^2 = 6^2 - 0^2 = 36$

So the shortest spacetime interval between A' and C' is to go thru B' (because A'B' and B'C' are light beams)!

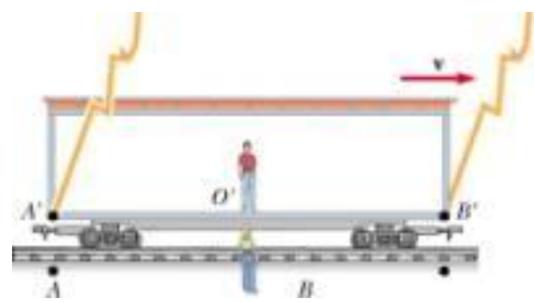


In the figure to the right, a particle has an initial velocity towards the left (at the bottom), but is experiencing a constant acceleration to the right. The dots along its worldline represent (from bottom to top) $\tau = -1$, $\tau = -1/2$, $\tau = 0$, $\tau = 1/2$, and $\tau = 1$. The path tangent vectors at each point show its velocity. Due to the nature of spacetime diagrams, the points do not appear to be evenly spaced nor do the vectors appear to be of equal lengths, but they are.

Vertical lines in spacetime diagrams means the object is not moving, straight angled lines means the object has a constant velocity, and curved lines means the object is experiencing acceleration. Horizontal lines are spacelike.

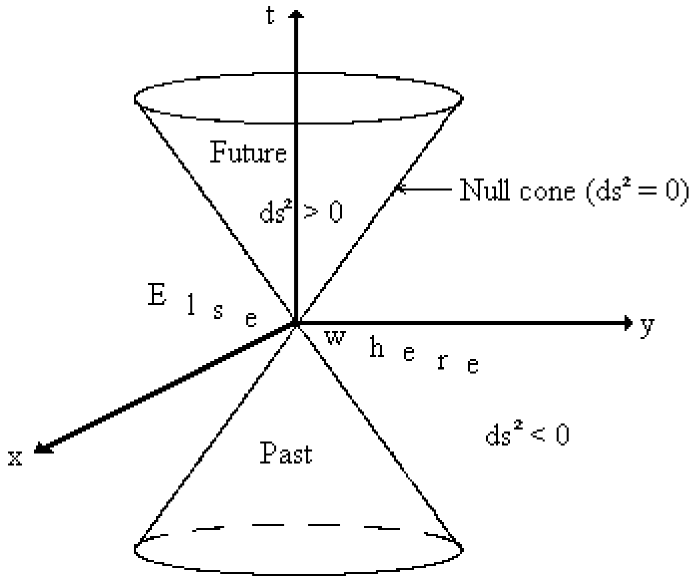
Simultaneity and Cause/Effect

Two events that happen in different locations that occur simultaneously in the reference frame of one observer may not appear to occur simultaneously in the reference frame of another observer. For example, consider two lightning bolts that strike both ends of a moving railroad car at the same time as seen by someone standing on the ground. → To someone standing in the middle of the moving car, the light from B reaches him first (because he is travelling towards it), so he would say that lightning strike B happened before A. So the concept of absolute simultaneity does not exist in special relativity.



But if event A *causes* event B, couldn't this lack of absolute simultaneity turn things around so that in some frame B appears to happen before A? Fortunately, no. If A and B are separated by a timelike interval (which would have to be the case for A to somehow affect the events at B), then A will appear to precede B in all frames of reference. The only conditions where an event B could appear to come after *or* before some event A are when A and B are separated by a *spacelike* interval, which means that no physical signal can pass between them, and so it is impossible for one of them to be the cause of the other.

Light Cones



In a spacetime diagram, the set of all null vectors (= all straight lines for which $ds^2 = 0$) at an event P of spacetime creates a **light cone** at that event. → The event P is at the origin of the axes (note that only 2 space dimensions are used). The light cone divides the spacetime around P into three different regions :

1. Every point <u>on</u> the light cone can be connected to P by a geodesic (straight) worldline of <u>zero length</u> (remember, this is not a normal "length"!). What this really means is dt=dx or t=x which is a 45° straight line in a 2-D spacetime graph. All curves of zero length (geodesic or not) are called **null curves**. Not every curve <u>on</u> the cone is a geodesic with zero length – for example, a spiral winding its way towards P is not.
2. Every point inside the "future" (where $t > t_P$) and "past" (where $t < t_P$) regions can be connected to P by a timelike curve. Events in the past cone could have influenced P, and P can potentially influence events within its future cone.
3. No point in the "elsewhere" region ($ds^2 < 0$) can be causally connected to P, because a signal happening there would have to travel faster than the speed of light to get to P (or vice-versa).

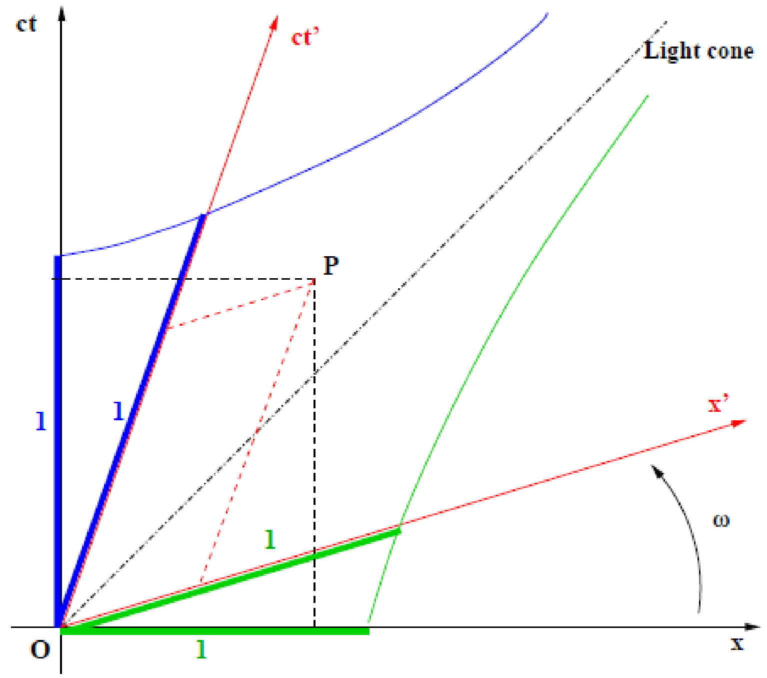
Another way to think of timelike, null, and spacelike : take any worldline through spacetime and look at the light cones at each point on it. If the next little bit of the curve is inside the light cone then that stretch of the curve is timelike; if it lies on the light cone then it is null (or lightlike); and if it is outside the light cone then it is spacelike.

NOTE : <u>**at**</u> the <u>*instant*</u> P happens, **NONE** of the rest of the universe is accessible, except for sound waves and light waves coming from the past! Just something to think about…

Minkowski Diagrams

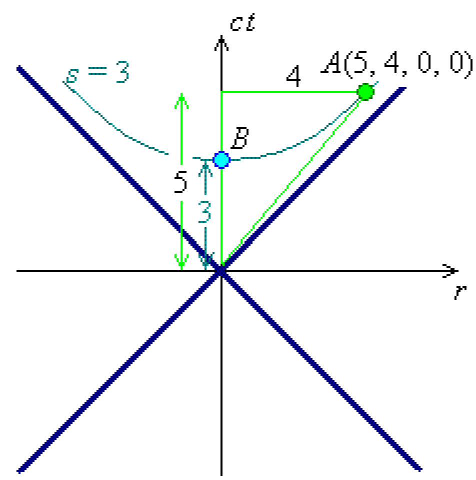
The terms "Minkowski diagram" and "spacetime diagram" are often used interchangeably, but here the term "Minkowski diagram" will be used only for spacetime diagrams that show a superposition of coordinate systems for two observers moving relative to each other with constant velocity. Their main purpose is to allow the space and time coordinates *x* and *t* used by one observer to be read off immediately as the coordinates *x′* and *t′* used by the other observer and vice versa. These diagrams allow a quantitative understanding of the corresponding phenomena like time dilation and length contraction without mathematical equations.

This figure → shows the coordinates of one observer (ct, x) who considers himself at rest, and the coordinates of another observer (ct', x') moving relative to him. The (ct', x') coordinates are not orthogonal to each other in the diagram, but they are orthogonal to each other in spacetime. They are skewed in the figure because they must remain symmetric around the 45° (v=c) line, because that line must remain the same for both frames! The blue hyperbola is the line $ds^2=1$ (for both frames), and the green hyperbola is $ds^2=-1$. These lines show how the scaling of the axes differs between frames (with lengths x=1 and ct=1 shown in bold). The figure shows how the two observers assign different (ct, x) values to the spacetime location of event P, but both agree on its length (not shown). The angle ω between the x axes describes the velocity *v* between the frames : v/c = tan(ω).



Example: Measuring distance in light-years and time in years, say some event *A* will occur five years from now at Alpha Centauri, 4 light-years away (see figure to the right). The spacetime separation of that event from here-and-now is

$$ds^2 = 5^2 - 4^2 = 25 - 16 = 9 \rightarrow ds = 3 \text{ years}$$

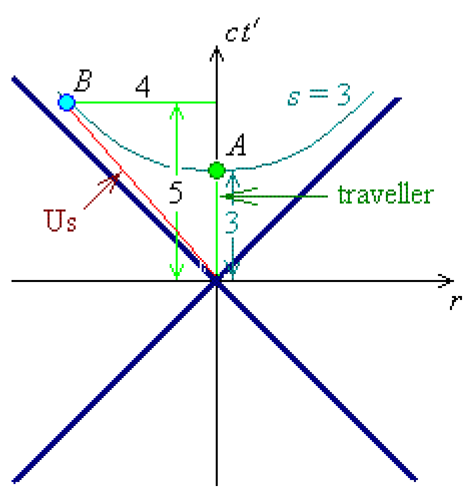


We can influence that event because light has enough time to reach that event ($ds^2 > 0$ → timelike). We could even reach that event ourselves, if we could travel at a speed of 80% of the speed of light. If, instead, we don't move for three years, then we will move up our time axis to the event B, which has the same *spacetime* separation from us (ds = 3 years) as event A, but occurs two years earlier, according to us.

Again, all events that have the same *spacetime* separation of 3 years from here-and-now trace a hyperbola, intersecting the time axis at 3 years (representing events that happen at the same point in space).

Any traveler who passes by us here-and-now with just the right constant velocity to reach the event four years away and five years in the future will need five years by our clock for the journey. However, the traveler's high speed causes the clocks on the traveler's spacecraft to run slow (compared to ours). The same journey requires only three years by the traveler's clock.

From the point of view of the traveler, they are "at rest". It is we, and their destination four light years away, that are rushing by at a significant fraction of the speed of light. The initial spatial distance between the traveler and the destination point is shorter in the traveler's frame of reference, some 2.4 years, (which requires three years to cover at 80% of the speed of light). On the traveler's own spacetime diagram, → the traveler is stationary and therefore simply moving up her time axis for three years, until meeting the event.



According to the traveler, we are rushing backward, with our clocks running slow. The traveler deduces that it takes us only three years on our clock, but five years on their clock, for us to reach event B. Note that the order in which events A and B occur is the other way around for the traveler! That's because A and B are separated by a spacelike interval (A can't get to B, even travelling at the speed of light).

Lorentz Transformations

Consider a particle moving with a constant velocity **v** in S (the "observer's" frame), and define S' so that its origin is always on the particle (the particle's "rest frame"). m_0 is the particle's mass in S', or its **rest-mass**. The coordinate system in S is [ct, x, y, z] and [ct', x', y', z'] in S'. To begin with, let the particle's velocity be along the x-axis in S. Then the relationship between the coordinates in S and S' is :

$t' = \gamma(t - xv/c^2)$ → $ct' = \gamma(ct - xv/c) = \gamma(ct - \beta x)$

$x' = \gamma(x - vt)$ → $x' = \gamma(x - \beta ct)$

$y' = y$

$z' = z$

Which in matrix form becomes

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

The matrix in this equation is called the Lorentz transformation matrix **L**, and is a type (1,1) tensor. This transformation applies to any 4-vector V, not just the coordinates. In coordinate notation, this becomes :

$$V'^u = L^u{}_v V^v$$

In order to transform a tensor, one transformation must be applied for every index. For example, to transform the metric g_{uv} into g_{ab} :

$$g_{ab} = L^u{}_a L^v{}_b \, g_{uv}$$

Or in matrix notation (where the order of terms matters) : $\mathbf{g}' = \mathbf{L}\,\mathbf{g}\,\mathbf{L}^T$

The length of every 4-vector must remain invariant under any Lorentz transformation, which means that observers in every inertial frame will agree on the length of a particular 4-vector. In addition, a **Lorentz scalar** is a quantity that remains invariant under Lorentz transformations, such as rest-mass and electric charge. Another property of the Lorentz transformation is that every inertial frame will agree whether a certain piece of a worldline is timelike, null, or spacelike.

Notice the signs of the non-zero terms in the **L** matrix above. Other possible transformations include :

- $\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ \gamma\beta & -\gamma \end{bmatrix}$ flips x axis: Space inversion
- $\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} = \begin{bmatrix} -\gamma & \gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}$ flips t : Time reversal

These reverse the orientations of space and time, and while they don't violate any laws of physics, we generally don't consider them.

Generalizing the velocity from along the x-axis to a **boost** in any direction, the Lorentz transformation is :

$$\mathbf{L} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix}$$

where γ and β use $|\mathbf{v}|$ and β_i uses v_i. This is the most common transformation in special relativity, since we are usually comparing inertial frames moving at different velocities to each other. Notice that there are space components "mixed up" with the time components.

The last main kind of transformations are **rotations** about the 3 spatial axes. These are often used, because a pure boost assumes that the axes of S' are parallel to S. For a rotation of θ degrees about the z-axis, the transformation tensor is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also **translate** the coordinate axes in any of the 4 directions, but in relativity this is not used too often. Note that time and space don't get "mixed up" for rotations and translations. The collection of boosts, rotations, and translations is known as the **Poincaré group** of transformations.

<u>Doing Physics in Special Relativity</u>

Consider a particle moving with a constant 3-velocity **v** (the space components) in S (the "observer's" frame), and define S' so that its origin is always on the particle (the particle's "rest frame"). m_0 is the particle's mass in S', or its **<u>rest-mass</u>**. The coordinate system is [ct, x, y, z].

The particle's position in S (its worldline) is given by : $x^a = (ct, x^1, x^2, x^3)$

Then the particle's **4-velocity** u^a at any point on the curve is a *path tangent vector* to its worldline, $dx^a/d\tau$:

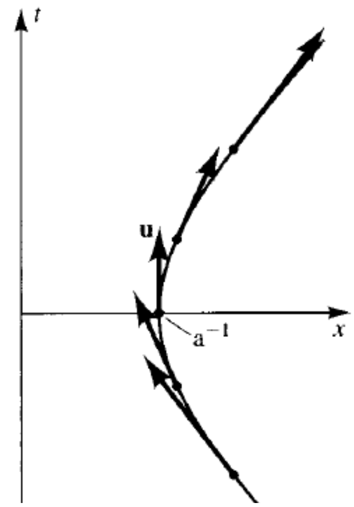
$$u^a = dx^a/d\tau = \gamma\, dx^a/dt = (\gamma c, \gamma v^1, \gamma v^2, \gamma v^3)$$

And the squared length of this vector is $u^0u^0 - u^iu^i = \gamma^2(c^2 - v^2) = c^2$ no matter what its "actual" velocity is, so it is invariant. And since $c^2 > 0$, it is also timelike. In the rest frame of the particle, its 4-velocity is (c,0,0,0). Note that not all 4-vectors add like "regular" vectors! As a simple example with 4-velocities, consider a train travelling east with velocity v as measured by someone standing on the ground. Someone inside the train throws a ball east with velocity u as measured by someone on the train. Newtonian physics says that the ball's velocity as measured by the person on the ground is v+u, but special relativity says the velocity will be $(v+u)/(1+vu/c^2)$. So when a spaceship travelling at v=0.5c fires a missile that travels at u=0.5c, someone outside the ship sees the missile travelling at 0.8c. But if both u and v are << c, the equation simplifies to v+u.

In addition, velocities measured <u>by an observer in the frame</u> are not 4-velocities ($u^i = dx^i/d\tau$), but the derivatives with respect to the local time ($v^i = dx^i/dt = u^i/u^t$).

The **4-acceleration** is what you would expect : $a^a = du^a/d\tau$ and is orthogonal to the velocity :

$\mathbf{a} \cdot \mathbf{u} = a^a u^a = (du^a/d\tau)u^a = d(u^a u^a)/d\tau = dc^2/d\tau = 0$



Since 4-velocity is timelike and 4-acceleration is orthogonal to it, 4-acceleration must be spacelike : in the figure to the right, the tangent velocity vectors are shown for equal time intervals. Since acceleration is $\Delta x/\Delta t$, the acceleration is the line you get from one arrowhead to the next. Since any line whose angle with respect to the x-axis is greater than ±45° is spacelike, it can be seen that all the 4-accelerations are spacelike.

The **4-momentum** is just *rest-mass* times 4-velocity :

$\mathbf{P} = m_0\mathbf{u} = (m_0\gamma c, m_0\gamma v^1, m_0\gamma v^2, m_0\gamma v^3) = (mc, mv^1, mv^2, mv^3)$

where $m = \gamma m_0$ is the *relativistic mass* of the particle.

The norm of this vector is $m^2c^2 - m^2v^2 = m^2(c^2 - v^2) = m_0^2\gamma^2 (c^2 - v^2) = m_0^2c^2$, so rest mass is an invariant scalar.

A common shorthand when creating 4-vectors is to use a scalar for the time component and a vector for the spatial components, as in : $\mathbf{P} = (mc, m\mathbf{u}) = \gamma(m_0c, m_0\mathbf{u}) = (\gamma m_0 c, \mathbf{p})$ where the 3-momentum $\mathbf{p} = \gamma m_0\mathbf{v}$

Note that the first term (the "time" term) is one *c* shy of being an energy, so **P** is often written as (E/c, **p**), and is then called the "energy-momentum 4-vector".

In the rest frame of the particle, its 4-momentum is $(m_0c,0,0,0)$, and its length is $m_0^2c^2$ (which it better be, since it is invariant) so $E/c=m_0c$ or $E=m_0c^2$. This is the energy from the mass of the particle when it is not moving, and probably the most famous equation in history.

And in general :

norm = $E^2/c^2 - p^2 = m_0^2c^2$

or $E^2 - p^2c^2 = m_0^2c^4$

so $E^2 = m_0^2c^4 + p^2c^2$ (this is the total "mass-energy" of a moving particle; NOTE : $p = \gamma m_0 v$)

If v=0 then p=0, and we get $E=m_0c^2$ again. Note that when $m_0=0$ (photons), E=pc or p=E/c which means even tho they are massless, photons have a momentum and can exert a pressure.

And the **kinetic energy** of a moving object is defined as

$T = E - E_{v=0} = (m_0^2c^4 + p^2c^2)^{-1/2} - m_0c^2 =$ <*after a lot of math*> $= m_0c^2(\gamma - 1)$

(remember that γ is a function of velocity so kinetic energy is a function of u)

We can approximate γ when $v \ll c$ by : $\gamma = 1 + v^2/(2c^2) + 3v^4/(8c^4) + \ldots$ which gives us

$T = m_0c^2(\gamma - 1) = m_0c^2(v^2/(2c^2) + 3v^4/(8c^4) + \ldots)$

Using only the first term (because the remaining terms are divided by c^2 or higher), $T = ½mv^2$, which is the result in classical physics.

Going back to the original equation for the total energy ($E=m_0\gamma c^2$), we can do the same thing to get

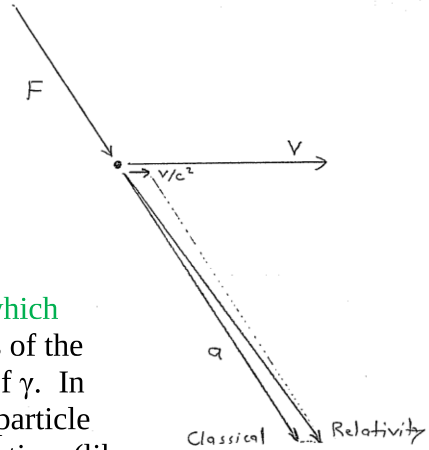
$E = m_0c^2 + ½m_0v^2 + 3v^4/(8c^2) + \ldots$

The first term, m_0c^2 is the rest-mass energy. The second term, $½m_0v^2$ is the (non-relativistic) kinetic energy. The remaining terms give minor relativistic corrections to the kinetic energy.

To find the **4-force**, we can start with Newton's laws which still hold when v << c, so in the rest frame of the particle : **f** = m**a** = d(m**v**)/dt = d**p**/dt. In special relativity this becomes, *using 3-vectors* :

$$\mathbf{f} = d\mathbf{p}/dt = d(m_0\gamma\mathbf{v})/dt = m_0[\gamma d\mathbf{v}/dt + \mathbf{v}d\gamma/dt] = m_0[\gamma\mathbf{a} + \mathbf{v}(d\gamma/dt)] = \gamma m_0[\mathbf{a} + \gamma^2\mathbf{v}(\mathbf{v}\cdot\mathbf{a})/c^2]$$

Note the extra term from the change in γ over time due to the acceleration (altho its contribution is very small due to $1/c^2$). Also note that the directions of the force and acceleration aren't in-line anymore due to the extra term. →



The natural extension is to take the derivative of the 4-momentum with respect to the proper time :

$$\mathbf{F} = d\mathbf{p}/d\tau = \gamma(d\mathbf{p}/dt) = \gamma(\mathbf{v}\cdot\mathbf{f}/c, \mathbf{f})$$

Note that **v**·**f** in the time component represents power (force·[distance/time]), which is also sometimes given as dE/dt ([force·distance]/time). Also, the components of the 4-force are not the same as the components of the 3-force due to the presence of γ. In the rest frame of the particle, $\mathbf{F} = (0,\mathbf{f}_0)$ where $\mathbf{f}_0$ is the Newtonian force on the particle in its frame. The time component is zero unless the *rest*-mass is changing over time (like a rocketship using up fuel as it flies, or an object gaining mass-energy from being heated).

Finally, consider the scalar value **v**·**f** :

$$\mathbf{v}\cdot\mathbf{f} = \mathbf{v}\cdot(d\mathbf{p}/d\tau) = \mathbf{v}\cdot[\mathbf{v}(dm_0/d\tau) + m_0(d\mathbf{v}/d\tau)] = \mathbf{v}\cdot\mathbf{v}(dm_0/d\tau) + m_0\mathbf{v}\cdot\mathbf{a} = c^2(dm_0/d\tau)$$

Thus in special relativity the action of a force can change the rest mass of an object! A force that doesn't change the rest mass is called a **pure force** and must satisfy **v**·**f** = 0, which means that the work done by the force goes completely into changing the kinetic energy of the object. For example, the force exerted on a charged particle by electric or magnetic fields.

A force that does not change the object's velocity at all is called "**heat-like**" or "**pure energy**" **source**, and increases the object's rest mass-energy by putting energy into its internal state, such as increasing its temperature.

Photons travel along null worldlines. Their **4-frequency** is tangent to (in-line with) their worldline, and is

$$\mathbf{k} = (\nu, \nu\mathbf{n})$$

where ν is the frequency and **n** is a unit vector in the direction the photon is travelling. Note that the norm is

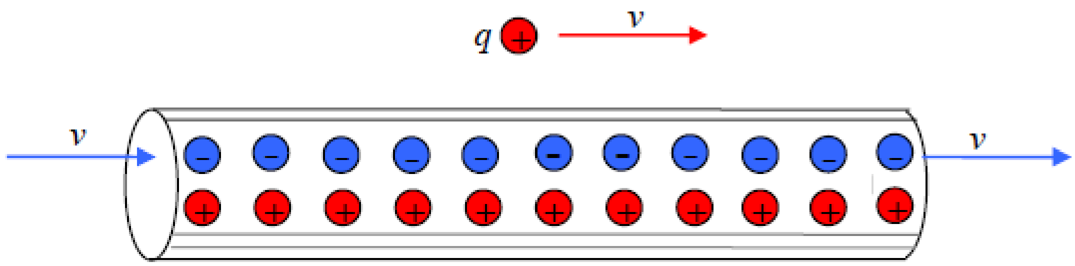
$$k^u k_u = \nu^2 - \nu^2 n^2 = 0$$

If an observer has velocity **u** and they measure the photon, they measure a frequency of : $u_a k^a$

A common source of confusion in relativity is the difference between quantities that are *conserved*, and quantities that are the same in all frames (*invariant*). Rest-mass and charge are the same in all frames, so in a closed system they are both invariant and conserved. Energy is always conserved, but it is not the same in all frames – the kinetic energy of your desk (which in your frame of reference is zero because it is not moving) is billions of joules from the sun's frame of reference (because of the desk's orbital rotational energy). Likewise, momentum is conserved but is not the same in all frames. While the energy E and the momentum **p** depend on the frame of reference in which they are measured, the quantity $E^2 - p^2c^2$ is invariant (because from above, it equals $m_0^2c^4$ which is the same for all observers). Velocity is neither conserved nor invariant.

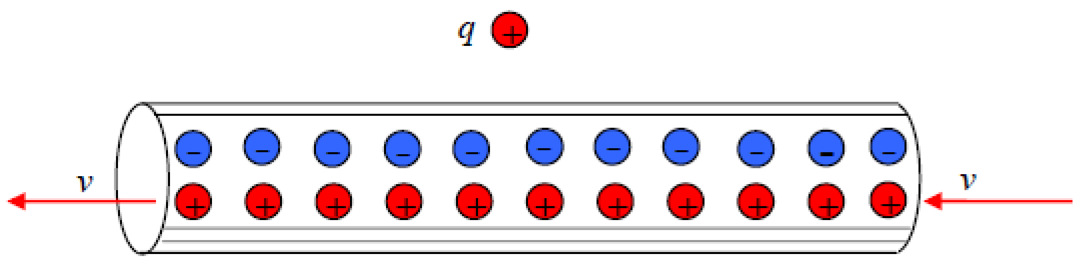
An Example

Suppose we have an infinitely long straight copper wire with electrons moving at speed *v* to the right and a charge density of $-\lambda$ coulombs per meter (*in their frame*), and a fixed background of positive charge (the copper ions the moving electrons came from) with a charge density of $+\lambda$ coulombs per meter (*in its frame*). There is a particle with charge +q outside the wire, at a perpendicular distance *r* from the wire, moving at the same exact velocity as the electrons. In the frame where the positive charges are at rest, we have :



What force does the particle feel? It feels no electric force since the wire is electrically neutral in this frame. But the moving negative charges create a (stationary) magnetic field of strength $B=\mu_0\lambda v/2\pi r$, and the particle will feel a magnetic force from the field, since it is moving thru it : $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. So the magnitude of the force is $F = q\mu_0\lambda v^2/2\pi r$.

Now look at the same system in a frame of reference moving along with the particle (and the electrons) :



What force does the particle feel? Now the "moving" positive charges create a (stationary) magnetic field, but the particle isn't moving thru it, so it feels no magnetic force. And at first glance, it also doesn't feel any net electric force because the positive and negative charge densities are equal. But, the charge density of the moving positive charges is <u>increased</u> by the Lorentz contraction due to their velocity to $\gamma\lambda \approx \lambda + \lambda v^2/2c^2$ (if $v << c$). On the other hand, in this frame the electrons are at rest, so their density has actually <u>decreased</u> by $\lambda v^2/2c^2$. The electric field from this non-zero charge density difference is $E = \lambda v^2/2\pi c^2\varepsilon_0 r$, and so the force is $F = qE = \lambda v^2/2\pi c^2\varepsilon_0 r = q\mu_0\lambda v^2/2\pi r$ (because $1/c^2 = \varepsilon_0\mu_0$), the same as before.

So a stationary charge in one frame creates an <u>electric field</u> in that frame; to a frame travelling by the charge (so the charge appears to be moving relative to the frame), it creates a <u>magnetic field</u>!